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Complex analysis examples 09

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2014-15/cx_ex_09.pdf]

If you want feedback from me on your treatment of these examples, please get your work to me by Monday, Feb 23, preferably as a PDF emailed to me.

[09.1] Prove that

$$\lim_{N\to+\infty} \prod_{n=1}^{N}(1+\frac{1}{n}) \;=\; +\infty \qquad \text{and} \qquad \lim_{N\to+\infty} \prod_{n=2}^{N}(1-\frac{1}{n}) \;=\; 0$$

[09.2] Following Euler, show that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges, by using the Euler product expansion of $\zeta(s)$ and considering $s \to 1^+$ along the real axis.

[09.3] Prove that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ does not vanish in $\mathrm{Re}(s) > 1$.

[09.4] Prove that $\Gamma(s)\cdot\Gamma(1-s) = \pi/\sin \pi s$, hence that $\Gamma(s)$ has no zeros, and $1/\Gamma(s)$ is entire.

[09.5] Prove that $\frac{1}{\Gamma(s)} = s\, e^{a+bs} \cdot \prod_{n=1}^{\infty}(1+\frac{s}{n})\, e^{-s/n}$ for some constants a, b.

[09.6] Let $d(n)$ be the *divisor function*, that is, the number of positive divisors of an integer n. Show that d is *weakly multiplicative* in the sense that $d(mn) = d(m)\cdot d(n)$ for m, n relatively prime, and that $d(p^\ell) = \ell + 1$ for p prime, and give some estimate on $d(n)$ adequate to show that $\sum_{n\ge 1} d(n)/n^s$ is absolutely convergent for $\mathrm{Re}(s)$ sufficiently large positive. Show that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} \;=\; \zeta(s)^2$$

[09.7] *(A variant Perron identity)* Show that, for $\sigma > 0$, a vertical path integral moving upward along the line $\mathrm{Re}(s) = \sigma$ evaluates to

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s(s+\theta)}\, ds \;=\; \begin{cases} \frac{1}{\theta}(1-X^{-\theta}) & (\text{for } X > 1) \\ 0 & (\text{for } 0 < X < 1) \end{cases} \qquad (\text{for } \theta > 0,\ \sigma > 0)$$

[09.8] In the Gaussian integers $\mathbb{Z}[i]$, there are 4 units $\pm 1, \pm i$. The *norm* is $N(m+in) = m^2 + n^2$. Show that the zeta function

$$\zeta_{\mathbb{Z}(i)}(s) \;=\; \frac{1}{\#\mathbb{Z}[i]} \sum_{0\ne m+in\in\mathbb{Z}[i]} \frac{1}{N(m+in)^s} \;=\; \frac{1}{4}\sum_{m,n \text{ not both } 0} \frac{1}{(m^2+n^2)^s}$$

has an analytic continuation and functional equation

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Z}[i]}(s) \;=\; \pi^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Z}[i]}(1-s)$$

by using

$$\theta(y)^2 \;=\; \Big(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\Big)^2 \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)y}$$