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Complex analysis midterm 01

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http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/midterm_discussion_01.pdf]

[01.1] Determine all values of $\left(\frac{1+i}{\sqrt{2}}\right)^i$.

By recognizing that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, that $e^{i\theta} = \cos\theta + i\sin\theta$, and the ambiguity that $e^{2\pi i n} = 1$ exactly for $n \in \mathbb{Z}$, we have

$$\frac{1+i}{\sqrt{2}} \;=\; e^{\frac{\pi i}{4} + 2\pi i n} \qquad \text{(exactly for } n \in \mathbb{Z})$$

Then

$$\left(\frac{1+i}{\sqrt{2}}\right)^i \;=\; e^{(\frac{\pi i}{4} + 2\pi i n)\cdot i} \;=\; e^{-\frac{\pi}{4} - 2\pi n} \qquad \text{(for all } n \in \mathbb{Z})$$

[01.2] Determine the Laurent expansion of $f(z) = 1/(1+z^2)^3$ in the annulus $1 < |z|$.

Expanding a geometric series after rearranging just a bit,

$$\frac{1}{1+z} \;=\; \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} \;=\; \frac{1}{z}\cdot\Big(1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\Big) \;=\; \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \ldots \;=\; \sum_{n=1}^{\infty} (-1)^n\, z^{-n}$$

Abel's theorem, adapted to Laurent series (!), assures us that we can differentiate termwise, here, *twice* to achieve our goal:

$$\frac{(-1)(-2)}{(1+z)^3} \;=\; \Big(\frac{d}{dz}\Big)^2 \sum_{n=1}^{\infty} (-1)^n\, z^{-n} \;=\; \sum_{n=1}^{\infty} (-1)^n\,(-n)(-n-1)\, z^{-n-2}$$

Replacing z by z^2 and dividing by 2,

$$\frac{1}{(1+z)^3} \;=\; \tfrac{1}{2}\sum_{n=1}^{\infty} (-1)^n\, n(n+1)\, z^{-2n-4} \qquad \text{(in } |z| > 1)$$

[01.3] Compute $\displaystyle\int_0^\infty \frac{x\,dx}{x^4+1}$.

We exploit the fact that the integrand transforms very simply upon replacing x by ix.

The integral is really the limit of the integrals $\int_0^R$ as $R \to +\infty$, with the same integrand. Add to this line segment the arc from R to iR, and then the integral along the segment from iR to 0, giving a closed curve γ_R. The integral along the arc is estimated by the trivial estimate:

$$\Big|\int_{R-\text{arc}} \frac{x\,dx}{x^4+1}\Big| \;\le\; (\text{length of arc}) \cdot \sup_{\text{on arc}} \Big|\frac{x}{x^4+1}\Big| \;=\; \frac{\pi R}{2}\cdot\frac{R}{(R-1)^4} \;\longrightarrow\; 0 \qquad (\text{as } R \to +\infty)$$

Parametrizing the segment from iR to 0 by $[0, R]$, the integral from iR to 0 is

$$\int_0^R \frac{(iR-it)\, d(iR-it)}{(iR-it)^4+1} \;=\; \int_0^R \frac{(R-t)\,dt}{(R-t)^4+1} \;=\; -\int_R^0 \frac{t\,dt}{t^4+1} \;=\; \int_0^R \frac{t\,dt}{t^4+1}$$

That is, this second line-segment integral is equal to the original. Thus,

$$2 \times \int_0^R \frac{x\;dx}{x^4+1} \;=\; \int_{\gamma_R} \frac{z\;dz}{z^4+1} - \int_{R-\mathrm{arc}} \frac{z\;dz}{z^4+1}$$

For $R > 1$, the integral around γ_R encloses just one singularity of the integrand, namely, at the primitive eighth root of unity $z_o = \zeta = e^{\pi i/4}$ lying in the first quadrant. Thus, by the Residue Theorem,

$$\int_0^\infty \frac{x\;dx}{x^4+1} \;=\; \tfrac{1}{2}\lim_R \int_{\gamma_R} \frac{z\;dz}{z^4+1} \;=\; \tfrac{1}{2}\lim_R 2\pi i \mathrm{Res}_{z=z_o} \frac{z\;dz}{z^4+1} \;=\; \tfrac{1}{2}\lim_R 2\pi i \frac{\zeta}{(\zeta-\zeta^3)(\zeta-\zeta^5)(\zeta-\zeta^7)}$$

$$= \pi i \cdot \frac{\zeta}{(\sqrt{2})(2\zeta)(i\sqrt{2})} \;=\; \frac{\pi}{(\sqrt{2})(2)(\sqrt{2})} \;=\; \frac{\pi}{4}$$

[01.4] Compute $\displaystyle\int_{-\infty}^\infty \frac{e^{itx}\;dx}{x^2+1}$ with real t.

The integral is the limit of $\int_{-R}^R$ of the same integrand, as $R \to +\infty$. Complete this line segment to a closed path γ_R by adding to it the arc from R to $-R$ through the upper half-plane. For $t \geq 0$, the exponential is bounded in the upper half-plane, and we estimate the integral over the arc by the trivial estimate:

$$\Big|\int_{R-\mathrm{arc}} \frac{e^{itx}\;dx}{x^2+1}\Big| \;\leq\; (\text{length of arc}) \cdot \sup_{\text{on arc}} \Big|\frac{e^{itx}}{x^2+1}\Big| \;=\; \pi R \cdot \frac{R}{(R-1)^2} \;\longrightarrow\; 0 \qquad (\text{as } R \to +\infty)$$

There is just one singularity inside γ_R for $R > 1$, namely, at $z = i$, so, by residues,

$$\int_{-\infty}^\infty \frac{e^{itx}\;dx}{x^2+1} \;=\; \lim_R \int_{\gamma_R} \frac{e^{itz}\;dz}{z^2+1} \;=\; \lim_R 2\pi i\,\mathrm{Res}_{z=i} \frac{e^{itz}}{z^2+1} \;=\; 2\pi i \frac{e^{-t}}{i-(-i)} \;=\; \pi e^{-t} \qquad (\text{for } t \geq 0)$$

For $t < 0$, the change of variables $x \to -x$ in the integral converts the integral to the $t \geq 0$ case, with $|t|$ in place of $t < 0$, and

$$\int_{-\infty}^\infty \frac{e^{itx}\;dx}{x^2+1} \;=\; \pi e^{-|t|}$$

[01.5] Compute $\displaystyle\frac{1}{1^2+1} + \frac{1}{2^2+1} + \frac{1}{3^2+1} + \ldots$

Use the auxiliary function $\frac{2\pi i}{e^{2\pi i z}-1}$, which we grant has singularities only at integers. We also grant that these are simple poles, with residue 1. For $R \in \frac{1}{2} + \mathbb{Z}$, let γ_R be the counter-clockwise path integral around a square of side $2R$ centered at 0, and consider

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{2\pi i}{e^{2\pi i z}-1} \cdot \frac{1}{z^2+1}\;dz$$

On one hand, granting that $\frac{2\pi i}{e^{2\pi i z}-1}$ is bounded away from its poles, the trivial estimate on path integrals shows that the integral over γ_R goes to 0 as $R \to +\infty$. Thus, by residues,

$$0 \;=\; \sum_{n\in\mathbb{Z}} \mathrm{Res}_{z=n} \frac{2\pi i}{e^{2\pi i z}-1} \cdot \frac{1}{z^2+1} \;+\; \sum_{\pm} \mathrm{Res}_{z=\pm i} \frac{2\pi i}{e^{2\pi i z}-1} \cdot \frac{1}{z^2+1}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} \;+\; \frac{2\pi i}{e^{2\pi i(i)}-1} \cdot \frac{1}{i-(-i)} \;+\; \frac{2\pi i}{e^{2\pi i(-i)}-1} \cdot \frac{1}{-i-i}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} \;+\; \frac{\pi}{e^{-2\pi}-1} \;-\; \frac{\pi}{e^{2\pi}-1}$$

Thus, subtracting the term for $n = 0$ and dividing by 2,

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} \;=\; \tfrac{1}{2}\cdot\Big(\frac{\pi}{e^{2\pi}-1} \;+\; \frac{\pi}{1-e^{-2\pi}} - 1\Big)$$

Returning to the details we granted ourselves to make things work:

The only singularities of $\frac{2\pi i}{e^{2\pi i z}-1}$ are where the denominator vanishes, which is at integers. The function $z \to e^{2\pi i z}$ is $\mathbb{Z}$-*periodic*, so determination of the residue at $z = 0$ determines all. Near $z = 0$,

$$\frac{2\pi i}{e^{2\pi i z}-1} \;=\; \frac{2\pi i}{(1 + 2\pi i z + (2\pi i z)^2/2 + \ldots) - 1} \;=\; \frac{2\pi i}{2\pi i z + (2\pi i z)^2/2 + \ldots} \;=\; \frac{1}{z + 2\pi i z^2 + \ldots}$$

$$=\; \frac{1}{z}\frac{1}{1 + (2\pi i z + \ldots)} \;=\; \frac{1}{z}\Big(1 - (2\pi i z + \ldots) + (2\pi i z + \ldots)^2 - \ldots\Big) \;=\; \frac{1}{z} - 2\pi i + ...$$

certifying that the residue is 1 at 0.

To check that $\frac{2\pi i}{e^{2\pi i z}-1}$ is *bounded* away from poles, first note that in $|\mathrm{Im}(z)| \geq 1$ it is bounded for simple reasons. In the region $|\mathrm{Im}(z)| \leq 1$ use periodicity to restrict $0 \leq \mathrm{Re}(s) \leq 1$. The region where $|z - 0| \geq \frac{1}{2}$, $|z - 1| \geq \frac{1}{2}$, and $|\mathrm{Im}(z)| \leq 1$ is *compact*, and $\frac{2\pi i}{e^{2\pi i z}-1}$ is continuous there, so is bounded.
