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Complex analysis midterm discussion02

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http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/midterm_discussion_02.pdf]

[02.1] Compute $\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-x^2}\, dx$ for real ξ.

The integral is the limit of finite integrals $\int_{-R}^{R} e^{i\xi x}\, e^{-x^2}\, dx$ as $R \to \infty$. Completing the square,

$$\int_{-R}^{R} e^{i\xi x}\, e^{-x^2}\, dx \;=\; e^{-\frac{\xi^2}{4}} \int_{-R}^{R} e^{-(x-\frac{i\xi}{2})^2}\, dx \;=\; e^{-\frac{\xi^2}{4}} \int_{-R+\frac{i\xi}{2}}^{R+\frac{i\xi}{2}} e^{-x^2}\, dx$$

To effectively shift the contour back to the real axis, first observe that Cauchy's theorem implies the vanishing of the integral around the rectangle with vertices $\pm R$ and $\pm R + \frac{i\xi}{2}$. The integrals on the vertical sides of this rectangle are estimated by

$$\Big| \int_{R}^{R+\frac{i\xi}{2}} e^{-z^2}\, dz \Big| \;\le\; \text{length} \cdot \sup \;\le\; \frac{|\xi|}{2} \cdot \sup_{z \text{ on } [R, R+\frac{i\xi}{2}]} e^{\mathrm{Re}\,(z^2)} \;=\; \frac{|\xi|}{2} e^{-R^2+\xi^2}$$

For fixed ξ, this goes to 0 as $R \to \infty$. The other vertical side is estimated essentially identically. Thus, the limits as $R \to \infty$ of the left-to-right integrals along the two horizontal sides of the rectangle are *equal*, giving

$$\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-x^2}\, dx \;=\; \lim_R e^{-\frac{\xi^2}{4}} \int_{-R+\frac{i\xi}{2}}^{R+\frac{i\xi}{2}} e^{-x^2}\, dx \;=\; \lim_R e^{-\frac{\xi^2}{4}} \int_{-R}^{R} e^{-x^2}\, dx \;=\; e^{-\frac{\xi^2}{4}} \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

Recall that the latter integral can be evaluated by squaring, converting to polar coordinates, and replacing r by $\sqrt{t}$:

$$\Big(\int_{-\infty}^{\infty} e^{-x^2}\, dx \Big)^2 \;=\; \int_0^{\infty} \int_0^{2\pi} e^{-r^2}\, d\theta\, r\, dr \;=\; 2\pi \int_0^{\infty} e^{-r^2}\, r\, dr$$

$$=\; 2\pi \int_0^{\infty} e^{-t}\, \sqrt{t}\, \frac{\frac{1}{2}\, dt}{\sqrt{t}} \;=\; \pi \int_0^{\infty} e^{-t}\, dt \;=\; \pi$$

[02.2] Compute $\int_0^{\infty} \frac{x^s\, dx}{x^2 - x + 1}$.

This converges absolutely for $-1 < \mathrm{Re}\,(s) < 1$. The computation will use $-1 < \mathrm{Re}\,(s) < 0$, and the identity principle assures us that the outcome is correct in the larger range.

Use a Hankel/keyhole contour H_ε with small $\varepsilon > 0$. That is, come in from $+\infty$ to ε, go counter-clockwise around a small circle of radius ε back to ε, and then back to $+\infty$. We want the real-valued $\log x$ for $x^s = e^{s \log x}$ on the part of the path from ε to $+\infty$, so on the earlier part of the path from ∞ to ε, the logarithm of x should be $\log x - 2\pi i$. As $\varepsilon \to 0$, the integral around the small circle goes to 0, and Cauchy's theorem implies that the integral over H_ε is *independent* of ε. Thus, the integral along H_ε is

$$\int_{H_\varepsilon} \frac{x^s\, dx}{x^2 - x + 1} \;=\; \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{x^s\, dx}{x^2 - x + 1} \;=\; (1 - e^{-2\pi i s}) \int_0^{\infty} \frac{x^s\, dx}{x^2 - x + 1}$$

On the other hand, the Hankel-contour integral is the limit of similar integrals to-and-from large positive R, rather than $+\infty$, as $R \to +\infty$. Adding a clockwise circle of radius R gives an integral over a closed path γ_R, which picks up $-2\pi i$ (negative sign because the path is clockwise) times the residues inside the path. The integral over the circle is estimated as usual by

$$\text{length} \cdot \sup \;\le\; 2\pi R \cdot \frac{R^{\mathrm{Re}\,(s)}}{(R-1)^2} \;\longrightarrow\; 0 \qquad\qquad (\text{for } \mathrm{Re}\,(s) < 0)$$

The only poles are at the zeros of the denominator, namely, the primitive sixth roots of unity. The arguments of these are obtained by starting with argument 0 on $[\varepsilon, R]$ and going clockwise, so the s^{th} powers of these sixth roots of unity are $e^{s\cdot(-\pi i/3)}$ and $e^{s\cdot(-5\pi i/3)}$. Thus, by residues,

$$\int_{H_\varepsilon} \frac{x^s\,dx}{x^2-x+1} \;=\; \lim_{R\to\infty} \int_{\gamma_R} \frac{x^s\,dx}{x^2-x+1} \;=\; -2\pi i\Big(\mathrm{Res}_{z=e^{-\pi i/3}} \frac{x^s}{x^2-x+1} + \mathrm{Res}_{z=e^{-5\pi i/3}} \frac{x^s}{x^2-x+1}\Big)$$

$$= \; -2\pi i\Big(\frac{e^{s\cdot(-\pi i/3)}}{e^{-\pi i/3}-e^{-5\pi i/3}} + \frac{e^{s\cdot(-5\pi i/3)}}{e^{-5\pi i/3}-e^{-\pi i/3}}\Big)$$

Thus,

$$\int_0^\infty \frac{x^s\,dx}{x^2-x+1} \;=\; \frac{-2\pi i}{1-e^{-2\pi i s}}\Big(\frac{e^{s\cdot(-\pi i/3)}}{e^{-\pi i/3}-e^{-5\pi i/3}} + \frac{e^{s\cdot(-5\pi i/3)}}{e^{-5\pi i/3}-e^{-\pi i/3}}\Big)$$

$$= \; \frac{-2\pi i}{-i\sqrt{3}}\,\frac{e^{-\pi i s/3}-e^{-5\pi i s/3}}{1-e^{-2\pi i s}} \;=\; \frac{2\pi}{\sqrt{3}}\,\frac{e^{\frac{2}{3}\pi i s}-e^{-\frac{2}{3}\pi i s}}{e^{\pi i s}-e^{-\pi i s}} \;=\; \frac{2\pi}{\sqrt{3}}\,\frac{\sin\frac{2}{3}\pi s}{\sin \pi s}$$

[02.3] Show that a holomorphic function f on a non-empty open set $U \subset \mathbb{C}$ such that $|f(z)| = 1$ for all $z \in U$ is necessarily constant.

Suppose f is *not* constant. Let φ be the inverse Cayley map $\varphi(z) = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(z) = \dfrac{z-i}{-iz+1}$. This maps the unit circle to the real line, apart from the point i, which is sent to infinity. The points in U where $f(z) = i$ cannot have an accumulation point in U, or else, by the identity principle, f is identically i. Thus, apart from the *discrete* set of points Z in U on which $f(z) = i$, the function $g(z) = (\varphi \circ f)(z)$ is real-valued and holomorphic. At $z \in U - Z$, for small real h, the difference quotients $\frac{g(z+h)-g(z)}{h}$ and $\frac{g(z+ih)-g(z)}{ih}$ are real and imaginary, respectively. As $h \to 0$, both limits are $g'(z_o)$, so both limits are 0. That is, $g' = 0$, and g is *constant*. Then $f = \varphi^{-1} \circ g$ is constant. ///

[02.4] Show that there is a holomorphic $f(z) = \sqrt[3]{z^4 - 1}$ near any point z_o with $z_o^4 \ne 1$. Determine the radius of convergence of the power series for $f(z)$ expanded at 0.

Recall that there is *a* holomorphic logarithm defined near z_o off the non-positive real axis $(-\infty, 0]$ by

$$\int_1^z \frac{dw}{w}$$

where the integration is along a straight line segment from 1 to z. Near z_o off the *positive* real axis $[0, +\infty)$, another logarithm can be defined by

$$\int_1^{-z} \frac{dw}{w} + \pi i$$

With $L(z)$ being either of these, we do have $e^{L(z)} = z$, since $L'(z) = 1/z$, $L(1) = 0$, and the second derivative of $e^{L(z)}$ vanishes:

$$\Big(e^{L(z)}\Big)'' \;=\; \Big(L'(z) \cdot e^{L(z)}\Big)' \;=\; \frac{-1}{z^2} \cdot e^{L(z)} + \Big(\frac{1}{z}\Big)^2 \cdot e^{L(z)} \;=\; 0$$

Thus, $(e^{\frac{1}{3}L(z)})^3 = z$ for z near any $z_o \neq 0$, by using one or the other of L_1 or L_2. Thus, for $z_o \neq \pm 1, \pm i$, there are holomorphic logarithms $L_1(z-1), L_2(z+1), L_3(z-i), L_4(z+i)$ for z near z_o, and

$$\Big(e^{\frac{1}{3}(L_1(z-1)+L_2(z+1)+L_3(z-i)+L_4(z+i))}\Big)^3 \;=\; (z-1)(z+1)(z-i)(z+i) \;=\; z^4-1$$

At $z_o = 0$, the power series for $L_1(z-1)$ converges absolutely on the largest open disk centered at 0 on which $L_1(z-1)$ is holomorphic. Since there is a holomorphic logarithm $L(z-1)$ on the half-plane $\text{Re}(z-1) < 0$ (for example), there certainly is a holomorphic logarithm on $|z| < 1$. Thus, the power series at $z_o = 0$ for $\log(z-1)$ converges at least on the open unit disk. The same applies to logarithms of $z+1$, $z-i$, and $z+i$. Thus, there is holomorphic

$$\sqrt[3]{z^4-1} \;=\; e^{\frac{1}{3}(L_1(z-1)+L_2(z+1)+L_3(z-i)+L_4(z+i))}$$

at least on the open unit disk. On the other hand, while this discussion shows that there *are* holomorphic $(z+1)^{1/3}$, $(z-i)^{1/3}$, and $(z+i)^{1/3}$ at $z = 1$, there is no holomorphic $(z-1)^{1/3}$ at $z = 1$. Among several ways to be sure of this, one way is to look at power series expansions:

$$\Big(c_o + c_1(z-1) + \ldots\Big)^3 \;=\; c_o^3 + 3c_o^2 c_1(z-1) + \ldots$$

For $c_o \neq 0$ this cannot be $z-1$, but for $c_o = 0$ the linear term of the cube is inevitably 0. Thus, the power series cannot converge at $z = 1$, so the radius of convergence is exactly 1. ///
