[03.1] **Give an explicit conformal map of the half-disk** \( \{ z = x + iy : |z| < 1, \ x > 0 \} \) **to the unit disk** \( \{ z : |z| < 1 \} \).

These are both non-degenerate bi-gons, so we know this can be accomplished by a composite of linear fractional transformations and power maps \( z \to z^\alpha \).

First, map one of the vertices \( \pm i \) to \( \infty \), and the other to 0, by \( z \to \frac{z+i}{z-i} \), for example. To determine the images of the sides, it suffices to track a third point on each, in addition to \( \pm i \). For the vertical straight line segment from \( -i \) to \( +i \) use the third point 0, which maps to \( -1 \). Thus, that segment maps to the ray along the negative real axis. For the half-circle, use third point 1, which maps to \( (1+i)/(1-i) = i \), so this side maps to the positive imaginary axis.

Rotate clockwise by \( \pi/2 \) radians, by multiplying by \( -i \), to put one edge on the **positive** real axis, so that the bi-gon becomes the interior of the first quadrant. Use \( z \to z^2 \) to map the first quadrant to the upper half-plane, and then the inverse Cayley map \( z \to \frac{z-i}{z+i} \) to map to the disk.

Altogether, this is

\[
\begin{align*}
  & z \rightarrow \frac{z+i}{z-i} \rightarrow -i \frac{z+i}{z-i} \rightarrow \left( -i \frac{z+i}{z-i} \right)^2 \rightarrow \frac{\left( -i \frac{z+i}{z-i} \right)^2 - i}{-i \left( -i \frac{z+i}{z-i} \right)^2 + i} \\
  & = \frac{-z^2 - 2iz + 1 - iz^2 - 2z + i}{iz^2 - 2z - i + z^2 - 2iz - 1} = \frac{-(1+i)z^2 - 2(1+i)z + (1+i)}{(1+i)z^2 - 2(1+i)z - (1+i)} \\
  & = \frac{-z^2 - 2iz + 1}{z^2 - 2z - 1}
\end{align*}
\]

mapping the half-disk to the disk. ///

[03.2] **Determine a finite set** \( S \subset \mathbb{C} \) **of points such that for** \( w_o \notin S \) **there is a holomorphic function** \( f(w) \) **near** \( w_o \) **such that** \( z = f(w) \) **gives a solution to the equation** \( z^5 - 5z - w = 0 \). (**Hint:** holomorphic inverse function theorem.)

In a relation \( F(z) = w \) with holomorphic \( F \), the holomorphic inverse function theorem can only fail at points \( z_o \) where \( F'(z_o) = 0 \). In the case at hand, \( F(z) = z^5 - 5z, \ F'(z) = 5(z^4 - 1), \) so the inverse function theorem can only fail at \( z_o = \pm 1, \pm i \). The corresponding values of \( w = F(z_o) \) are

\[
w_o = z_0^5 - 5z_o = z_0(z_0^4 - 1) + 4z_o = 4z_o = \pm 4, \pm 4i \quad \text{for} \ z_o^4 = 1
\]

Thus, excluding \( S = \{ \pm 4, \pm 4i \} \) ensures a local holomorphic inverse. ///

[03.3] **Show that** \( f(z) = e^{iz} - z \) **has at least one complex zero.**

One approach is by the **argument principle**: the net change of the argument of \( f \) around a large-enough box, with vertices \( \pm T \pm iT \), is \( 2\pi \times \text{the number of zeros inside} \) (assuming that \( T \) is adjusted so that there are no zeros exactly on the rectangle: this adjustment is possible, by the identity principle).
Along the top side, \(|e^{(x+iT)}| = e^{-T}\), which is (much!) smaller than \(|x+iT|\) for \(T \geq 1\), for example. Thus, along that top edge, the argument of \(e^{iz} - z\) stays within \(\pi/2\) of that of \(z = x+iT\). Thus, while \(\arg z\) goes from \(\pi/4\) to \(3\pi/4\) as \(z = x+iT\) goes from \(T+iT\) to \(-T+iT\), the argument of \(e^{iz} - z\) can at most have net change

\[
\left(\frac{3\pi}{4} + \frac{\pi}{2}\right) - \left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{3\pi}{2}
\]

and at least by

\[
\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) - \left(\frac{\pi}{4} + \frac{\pi}{2}\right) = -\frac{\pi}{2}
\]

In any case, it is \(O(1)\), in Landau’s notation.

Along the bottom side, \(|e^{(x+iT)}| = e^T\), which is (much!) larger than \(|\pm T \pm iT|\) for \(T \geq 6\), for example. Thus, along the bottom edge, the argument of \(e^{iz} - z\) stays within \(\pi/2\) of that of \(e^{iz}\). Thus, while \(\arg e^{i(x-iT)} = x\) goes from \(-T\) to \(+T\), the argument of \(e^{iz} - z\) changes at most by

\[
\left(T + \frac{\pi}{2}\right) - \left(-T - \frac{\pi}{2}\right) = 2T + O(1)
\]

and at least by

\[
\left(T - \frac{\pi}{2}\right) - \left(-T + \frac{\pi}{2}\right) = 2T + O(1)
\]

Along the vertical sides, use the trick that the absolute value of the net change in argument is at most \(2\pi(q+1)\) where \(q\) is the number of times the real part vanishes. Further, slightly adjust \(T\) so that \(\cos T = 0\), so that

\[
\text{Re}(e^{i(T \pm iy)} - (T \pm iy)) = \cos T - T = -T
\]

That is, the real part does not vanish at all along the vertical edges, so the net changes are bounded by \(\pm 2\pi = O(1)\).

Putting these together, for large-enough \(T\), the net change in the argument around the \(\pm T \pm iT\) rectangle is \(2T + O(1)\), so the number of zeros inside is \(\frac{2}{\pi} + O(1)\). For large-enough \(T\), this is greater than 1, so there is at least one zero inside.

///

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