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Complex analysis midterm discussion 05

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/midterm_discussion_05.pdf]

[05.1] Exhibit a change of variables so that

$$\int_a^b \frac{dx}{\sqrt{x^3-1}} \;=\; \int_A^B \frac{dy}{\sqrt{\text{quartic in } y}}$$

Letting $x = gy$ for any linear fractional transformation g that moves ∞ to a finite point, such as $x = y/(y-1)$. For this example, this gives $dx = \frac{-dy}{(y-1)^2}$, and then

$$\int \frac{dx}{\sqrt{x^3-1}} \;=\; \int \frac{\frac{-dy}{(y-1)^2}}{\sqrt{\left(\frac{y}{y-1}\right)^3 - 1}} \;=\; -\int \frac{dy}{\sqrt{-y^3+4y^3-6y^2+4y-1+y(y-1)}}$$

The limits can be rearranged, etc. ///

[05.2] For lattice $\Lambda \subset \mathbb{C}$, express $\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^5}$ in terms of $\wp(z)$ and $\wp'(z)$. Do not worry about explicit determination of constants, although explication of them would earn extra credit.

Let $f(z)$ be the given function. In fact, visibly, it is $-1/(2\cdot 3\cdot 4)$ times the third derivative of $\wp(z)$. It is *odd*. The Laurent expansion of the even function $\wp(z)$ at 0 is of the form

$$\wp(z) \;=\; \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \ldots$$

with vanishing constant due to the convergence trick. Thus,

$$\wp'(z) \;=\; \frac{-2}{z^3} + 2a_2 z + 4a_4 z^3 + \ldots$$

and

$$f(z) \;=\; \frac{-1}{2\cdot 3\cdot 4}\cdot\Big(\frac{-2\cdot 3\cdot 4}{z^5} + (2\cdot 3\cdot 4)a_4 z + \ldots\Big) \;=\; \frac{1}{z^5} + a_4 z + \ldots$$

Since $f(z)$ has no poles except on Λ, if we can cancel these poles by subtracting a polynomial in $\wp(z)$ and $\wp'(z)$, what is left must be a *constant*, by Liouville. To get rid of the $1/z^5$ term, subtract a multiple of $\wp(z)\cdot\wp'(z)$. The latter product is

$$\wp(z)\cdot\wp'(z) \;=\; \frac{-2}{z^5} + \frac{2a_2 - 2a_2}{z} + (4a_4 + 2a_2^2 - 2a_4)z + \ldots \;=\; \frac{-2}{z^5} + (2a_4 + 2a_2^2)z + \ldots$$

Thus,

$$f(z) + \tfrac{1}{2}\wp(z)\wp'(z) \;=\; (a_4 z + \ldots) + ((a_4 + a_2^2)z + \ldots) \;=\; O(z)$$

Thus, $f(z) + \frac{1}{2}\wp(z)\wp'(z)$ is a holomorphic elliptic function, so is constant, but vanishes at 0, so is 0. Further examination of the coefficients that appear is not necessary. ///

[05.3] Let ω_1, ω_2 be a basis for a lattice $\Lambda \subset \mathbb{C}$. Express $\wp(z + \frac{\omega_1}{2})$ as a rational function of $\wp(z)$. Do not worry about explicit determination of constants, although explication of them would earn extra credit.

The function $f(z) = \wp(z + \frac{\omega_1}{2})$ is still an *even* elliptic function, for the same lattice, and with double poles at $\Lambda + \frac{\omega_1}{2}$. Since the number of poles is the number of zeros, f has exactly two zeros, z_1, z_2 (which for some lattices will coincide), and by even-ness have the symmetry $z_2 = -z_1 \bmod \Lambda$.

The possibility that $z_1 \in \Lambda$ needs separate treatment.

First, treat $z_1 \notin \Lambda$. The function $\wp(z) - \wp(z_1)$ vanishes at z_1 and $-z_1$, and/or vanishes doubly for $z_1 = -z_1 \bmod \Lambda$. Since the number of zeros is the number of poles, these are the *only* zeros of $\wp(z) - \wp(z_1)$, and

$$\frac{\wp(z + \frac{\omega_1}{2})}{\wp(z) - \wp(z_1)}$$

has no zeros *off* Λ, and no new poles have been introduced. To get rid of the (double) poles off Λ, namely, at $\Lambda + \frac{\omega_1}{2}$, multiply by $\wp(z) - \wp(\frac{\omega_1}{2})$, which has a double zero at $\wp(\frac{\omega_1}{2})$, and no other zeros or poles off Λ. Thus,

$$\frac{\wp(z + \frac{\omega_1}{2})}{\wp(z) - \wp(z_1)} \cdot \Big(\wp(z) - \wp(\frac{\omega_1}{2})\Big)$$

has no zeros or poles off Λ. Since the number of zeros is the number of poles, that number must be 0, so the latter expression is *constant*, and

$$\wp(z + \frac{\omega_1}{2}) \;=\; \text{const} \times \frac{\wp(z) - \wp(z_1)}{\wp(z) - \wp(\frac{\omega_1}{2})} \qquad\qquad (\text{for } \wp(\tfrac{\omega_1}{2}) \neq 0)$$

The case that the zero z_1 of $\wp(z + \frac{\omega_1}{2})$ is in Λ is $\wp(\frac{\omega_1}{2}) = 0$, so $\wp(z)$ itself has a double zero at $\omega_1/2$ (and no others, by the same counting argument). That is, $\wp(z + \frac{\omega_1}{2})$ has double pole at $\omega_1/2$ and double zero at 0, while $\wp(z)$ has the opposite, Thus, in this case

$$\wp(z + \frac{\omega_1}{2}) \;=\; \text{const} \times \frac{1}{\wp(z)} \qquad\qquad (\text{for } \wp(\tfrac{\omega_1}{2}) = 0)$$

To determine the constant in the first (general) case, look at the power series expansion near $z = 0$, to find

$$\wp(z + \frac{\omega_1}{2}) \;=\; \wp(\frac{\omega_1}{2}) \cdot \frac{\wp(z) - \wp(z_1)}{\wp(z) - \wp(\omega_1)} \qquad\qquad (\text{for } \wp(\tfrac{\omega_1}{2}) \neq 0)$$

In the special case $\wp(\frac{\omega_1}{2}) = 0$, because all zeros and poles are already accounted for, $\wp(\frac{\omega_2}{2}) \neq 0$ and $\wp(\frac{\omega_1+\omega_2}{2}) \neq 0$, so we can evaluate at $z = \omega_2/2$ to determine the constant, and

$$\wp(z + \frac{\omega_1}{2}) \;=\; \frac{\wp(\frac{\omega_2}{2})}{\wp(\frac{\omega_1+\omega_2}{2})} \times \frac{1}{\wp(z)}$$

The values $\wp(\frac{\omega_1}{2})$ can be somewhat further elaborated in terms of expressions resembling Eisenstein series, if desired. ///
