(*May 1, 2015*)

Complex analysis midterm discussion 06

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/examples_2014-15/midterm_discussion_06.pdf]

[06.1] What is the genus of the projective curve arising from $y^2 = x^5 - 1$?

This is a *hyper-elliptic curve*, that is, of the form $y^2 =$ square-free polynomial in x: to see that $x^5 - 1$ is square-free, observe that it and its derivative $5x^4$ have no common factor. Thus, with d the degree of the polynomial in x, apply the specialized formula derived from the general Riemann-Hurwitz formula:

$$\text{genus of hyperelliptic curve} \;=\; \begin{cases} \frac{d-1}{2} & \text{for } d \text{ odd} \\ \frac{d-2}{2} & \text{for } d \text{ even} \end{cases}$$

to obtain

$$\text{genus} \;=\; \frac{5-1}{2} \;=\; 2$$

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[06.2] What is the genus of the projective curve arising from $y^5 = x^5 - 1$?

We need to use a more general version of Riemann-Hurwitz, which for an n-fold ramified covering $Y \to X$ is

$$2 - 2g_Y \;=\; n \cdot (2 - 2g_X) - \sum_{y_i} (e_i - 1)$$

where the sum is over ramified points y_i with ramification index e_i, and g_X, g_Y are the genuses.

Near $x \in \mathbb{C}$, for $x^5 - 1 \neq 0$ there are five distinct holomorphic fifth roots y, so there is no ramification.

Near $x \in \mathbb{C}$ such that $x^5 - 1 = 0$, we can easily invoke Newton polygons to see that there is a unique, *totally* ramified y over such x: in fact, letting ω be a primitive fifth root of 1, the polynomial

$$\sum_i c_i y^i \;=\; y^5 - (x-1)(x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4)$$

meets Eisenstein's criterion for each of the primes $x - \omega^i$ in $\mathbb{C}[x]$. That is, each of the corresponding Newton polygons (convex hull of points $(j, \text{ord}_{x-x_j} c_j)$ has *rise* 1 and *run* 5.

To examine ramification at infinity, replace y by $1/y$ and x by $1/x$: $(1/y)^5 = (1/x)^5 - 1$, or $x^5 = y^5 - x^5 y^5$, or

$$y^5 \;=\; \frac{x^5}{1 - x^5} \qquad \text{(near } x = 0\text{)}$$

The denominator has 5 distinct holomorphic fifth roots near $x = 0$, and x^5 has 5 distinct holomorphic fifth roots near $x = 0$, namely, $\omega^i x$ for $i = 0, 1, \ldots, 4$. Thus, there is *not* ramification at ∞. Applying Riemann-Hurwitz,

$$2 - 2g \;=\; 5 \cdot (2 - 2 \cdot 0) - 5 \cdot (5 - 1)$$

so

$$g \;=\; 1 - 5 + 10 \;=\; 6$$

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[06.3] Determine the points $x \in \mathbb{C}$ over which the curve $y^7 + 7xy + x^3 = 0$ has non-trivial ramification.

Considering that polynomial as an element $f(y)$ of $\mathbb{C}(x)[y]$, we use the Euclidean algorithm to compute the *gcd* of $f(y)$ and $f'(y)$ as an element of $\mathbb{C}(x)$, and then determine x_o such that either that *gcd* vanishes at x_o, or the Euclidean algorithm degenerates at x_o.

First, $f'(y) = 7y^6 + 7x$. Then $f(y) - \frac{y}{7} f'(y) = 6xy + x^3$. If $x = 0$, this is already 0, so $x_o = 0$ is a point-of-interest (which is visible at the outset). Noting this, we will divide-with-remainder $f'(y)$ by $\frac{1}{6x}(6xy + x^3) = y + \frac{x^2}{6}$. Dividing-with-remainder $f'(y)$ by $y - a$ gives remainder $f'(a)$, so the remainder here is

$$\text{gcd} \;=\; \text{remainder} \;=\; f'(\frac{x^2}{6}) \;=\; 7(\frac{x^2}{6})^6 + 7x \;=\; \frac{7}{6^6} \cdot (x^{12} + 6^6 x)$$

Thus, in addition to 0, there is possible ramification above the 11^{th} roots of -6^6. ///

Although the question didn't ask for it, the Newton polygon easily and unequivocally determines the ramification over 0: the segment from $(5-5,0) = (0,0)$ to $(5-1,1) = (4,1)$ has *rise* 1 and *run* 4, indicating a point with ramification index 4. The segment from $(5-1,1) = (4,1)$ to $(5-0,3) = (5,3)$ has rise 2 and run 1, indicating an *unramified* point above $x = 0$.

Slightly more subtly, for x an 11^{th} root of -6^6, the remainder upon dividing-with-remainder $f(y)$ by $f'(y)$ is linear in y. The vanishing of the remainder in the *next* step of the Euclidean algorithm shows that this linear polynomial is the *gcd* of $f(y)$ and $f'(y)$. That is, exactly one linear factor (in a polynomial ring with coefficients in an extension field $\mathbb{C}((x^{1/n}))$) appears twice in $f(y)$, and no other appears with multiplicity more than 1. Thus, at each of the 11^{th} roots of -6^6, there are 5 unramified points, and a single point with ramification index 2. ///

[06.4] What is the nature of the ramification of the curve $y^5 + xy^2 + x^6 = 0$ above a neighborhood of $x = 0$?

The Newton polygon has vertices $(5-5,0) = (0,0)$, $(5-2,1) = (3,1)$, and $(5,6)$. The segment from the first to the second has *rise* 1 and *run* 3, so indicates a point lying-over with ramification index 3. The segment from the second to the third has relatively-prime rise 5 and run 2, so slope $5/2$ indicates an over-lying point with ramification 2. In summary, there is a point with ramification index 3, and a point with ramification index 2. ///
