02.1 Parametrize counter-clockwise a circle $\gamma$ of radius $r > 0$ centered at $z_0$, and directly compute $\int_\gamma (z - z_0)^n \, dz$ for all positive and negative integers $n$.

**Discussion:** Such a path can be parametrized as $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$. Then

$$
\int_\gamma (z - z_0)^n \, dz = \int_0^{2\pi} (re^{it})^n \, d(re^{it}) = \int_0^{2\pi} (re^{it})^n ire^{it} \, dt
$$

$$
= ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt = \begin{cases}
2\pi i & \text{for } n = -1 \\
0 & \text{for } n \neq -1
\end{cases}
$$

as required.

02.2 Of what rational function is $\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)z^n$ the power series expansion at 0?

**Discussion:** If the power of $z$ were $z^{n-4}$, it might be even more obvious that the power series has something to do with the fourth derivative of $\sum z^n = \frac{1}{1-z}$. Yes, by an easy part of a theorem of Abel (and others), we can differentiate termwise to correctly find derivatives of power series within their radius of convergence. So

$$
\sum n(n-1)(n-2)(n-3)z^n = z^4 \cdot \left( \frac{d}{dz} \right)^4 \sum z^n = z^4 \cdot \left( \frac{d}{dz} \right)^4 \frac{1}{1-z} = z^4 \cdot \frac{4!}{(1-z)^5}
$$

as expected.

02.3 Determine the Laurent expansions of $\frac{1}{1-z}$ in $|z| < 1$, and in $|z| > 1$. Observe that these two have no common region of convergence.

**Discussion:** When $|z| < 1$, we have the iconic geometric series

$$
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n
$$

When $|z| > 1$,

$$
\frac{1}{1-z} = \frac{1}{z} \cdot \frac{1}{z-1} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = -\frac{1}{z} \cdot \sum_{n=0}^{\infty} (\frac{1}{z})^n = -\sum_{n=0}^{\infty} (\frac{1}{z})^{n+1}
$$

as required.

02.4 Using only geometric series expansions, determine the Laurent expansion of $f(z) = 1/(z-1)(z-2)$ in the annulus $1 < |z| < 2$, and also in the annulus $|z| > 2$.

**Discussion:** By partial fractions, for $1 < |z| < 2$, expanding geometric series,

$$
\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{1 - \frac{2}{z}} - \frac{1}{1 - \frac{1}{z}} = -\frac{1}{z} \left( 1 + \frac{2}{z} + \left( \frac{2}{z} \right)^2 + \ldots \right) - \frac{1}{z} \left( 1 + \frac{1}{z} + \left( \frac{1}{z} \right)^2 + \ldots \right)
$$
\[ \frac{1}{(z - 1)(z - 2)} = \frac{1}{z - 2} - \frac{1}{z - 1} = \frac{1}{z} \left( \frac{1}{1 - \frac{2}{z}} - \frac{1}{1 - \frac{1}{z}} \right) = \frac{1}{z} \left( \frac{1}{1 + \frac{2}{z}} + \left( \frac{1}{z} \right)^2 + \ldots \right) - \frac{1}{z} \left( \frac{1}{1 + \frac{1}{z}} + \left( \frac{1}{z} \right)^2 + \ldots \right) \]

\[ = \sum_{n=1}^{\infty} (2n-1) z^{-n} = \sum_{n=2}^{\infty} (2n-1) z^{-n} \quad \text{(in the annulus} \ 1 < |z| < 2) \]

as required. ///

**[02.5]** Determine the Laurent expansion of \( f(z) = 1/(z - 1)^3 \) in the annulus \( |z| > 1 \), and in the annulus \( |z - 1| > 0 \).

**Discussion:** In \(|z| > 1\),

\[ \frac{1}{z - 1} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \frac{1}{z} \left( 1 + \frac{1}{z} + \left( \frac{1}{z} \right)^2 + \ldots \right) = \frac{1}{z} + \left( \frac{1}{z} \right)^2 + \ldots \]

Differentiating termwise two times gives

\[ \frac{(-1)(-2)}{(z - 1)^3} = \frac{(-1)(-2)}{z^3} + \frac{(-2)(-3)}{z^4} + \ldots + \frac{(-n)(-n-1)}{z^{n+2}} + \ldots \]

which simplifies to

\[ \frac{1}{(z - 1)^3} = \frac{1}{z^3} + \frac{2 \cdot 3/5}{z^4} + \ldots + \frac{n(n+1)/2}{z^{n+2}} + \ldots \]

In the annulus \(|z - 1| > 0\), the given expression \( f(z) = (z - 1)^{-3} \) is already the Laurent expansion. ///

**[02.6]** Show that an entire function \( f \) satisfying \(|f(z)| \leq C \cdot (1 + |z|)^{1/2}\) for some constant \( C \) is constant.

**Discussion:** This is a variant of Liouville’s theorem and its proof. Since \( f \) is entire, its power series at 0

\[ f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} \cdot z^n \]

converges for all \( z \) and is equal to \( f(z) \). For \( 0 \leq n \in \mathbb{Z} \), use Cauchy’s formula for derivatives, integrating counter-clockwise over a circle \( \gamma_R \), of radius \( R \), centered at 0:

\[ f^{(n)}(0) = \frac{(n-1)!}{2\pi i} \int_{\gamma_R} \frac{f(w) \, dw}{(w - z)^{n+1}} \]

The easy/trivial estimate on the absolute value of the integral is that it is bounded by

\[ \text{length } \gamma_R \times \text{max of integrand on } \gamma_R \leq 2\pi R \times \frac{C \cdot (1 + |R|)^{1/2}}{R^{n+1}} \leq 2\pi R \times \frac{C \cdot (2R)^{1/2}}{R^{n+1}} \quad (\text{for } R \geq 1) \]

which is dominated by \( R^{3/2-(n+1)} \). For \( n \geq 1 \), this goes to 0 as \( R \to +\infty \). Thus, all the power series coefficients but the \( 0^{th} \) are 0, and \( f \) is a constant. ///

**[02.7]** Show that an entire function \( f \) satisfying \(|f(z)| \leq C \cdot (1 + |z|)^r\) for some \( 0 \leq r \in \mathbb{R} \), and for some constant \( C \), is a polynomial of degree at most \( r \). (Yes, degrees of not-identically-zero polynomials are non-negative integers.)
Discussion: Another variant of Liouville’s theorem. The argument basically recopies the previous, with a small difference at the end. Since $f$ is entire, its power series at 0

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} \cdot z^n$$

converges for all $z$ and is equal to $f(z)$. Use Cauchy’s formula for derivatives, integrating counter-clockwise over a circle $\gamma_R$, of radius $R$, centered at 0:

$$f^{(n)}(0) = \frac{(n-1)!}{2\pi i} \int_{\gamma_R} \frac{f(w) \, dw}{(w-z)^{n+1}}$$

The easy/trivial estimate on the absolute value of the integral is that it is bounded by

$$\text{length } \gamma_R \times \text{max of integrand on } \gamma_R \leq 2\pi R \times \frac{C \cdot (1 + |R|)^r}{R^{n+1}} \leq 2\pi R \times \frac{C \cdot (2R)^r}{R^{n+1}} \quad (\text{for } R \geq 1)$$

which is dominated by $R^{(r+1)-(n+1)}$. For $n > r$, this goes to 0 as $R \to +\infty$. Thus, all the coefficients of $z^n$ for $n > r$ are 0, and $f$ is a polynomial of degree at most $r$. ///

[02.8] Show that an entire function $f$ satisfying $|f(z)| \leq C \cdot \log(1 + |z|)$ for some constant $C$ is constant.

Discussion: Yet another variant of Liouville’s theorem...

[02.9] Compute $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$ without using arctangent.

Discussion: This is an iconic residue-theorem computation, and is (also) useful in practice. First, basically from the definition of path integral, the calculus-style integral is equal to the corresponding integral over the real line, and we acknowledge this by using variable $z$ instead of the usually-real variable $x$:

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{\infty} \frac{dz}{1 + z^2}$$

The parametrization can be by the real line itself, by $t \to t$ for $t \in \mathbb{R}$. As usual, infinite integrals are limits of finite ones, so these integrals are

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dz}{1 + z^2}$$

As usual, we want to close up the path of integration by adding an auxiliary arc, depending on $R$, which as $R \to +\infty$ goes to 0. Here, the auxiliary half-circle arc $\alpha_R$ parametrized by $t \to Re^{it}$ for $t \in [0, 2\pi]$ closes up the integral on $[-R, R]$, to a closed path $\gamma_R$. By residues, for any $R > 1$ (so that the pole of $1/(1 + z^2)$ at $z = i$ is inside the path), since there are no poles inside $\gamma_R$ except $i$,

$$\int_{\gamma_R} \frac{dz}{1 + z^2} = 2\pi i \text{Res}_{z=i} \frac{1}{1 + z^2} = 2\pi i \text{Res}_{z=i} \frac{1}{z - i} \cdot \frac{1}{z + i}$$

At this point, it is good to observe a fairly general useful way to evaluate residues: for $f$ holomorphic on a neighborhood of $z_o$, the residue of $f(z)/(z - z_o)$ at $z_o$ is $f(z_o)$. This is proven by obtaining the Laurent series of $f(z)/(z - z_o)$ at $z_o$ by dividing the power series of $f(z)$ at $z_o$ by $z - z_o$. Thus,

$$\int_{\gamma_R} \frac{dz}{1 + z^2} = 2\pi i \frac{1}{i + i} = \pi$$
The trivial estimate on the integral over the auxiliary arc gives

\[ \left| \int_{\alpha_R} \frac{dz}{(z-i)(z+i)} \right| = \text{length} \times \text{maximum on the arc} \leq \pi R \times \frac{1}{(R-1)^2} \longrightarrow 0 \quad \text{(as } R \rightarrow +\infty) \]

That is, as desired, in the limit, the integral over the auxiliary arc goes to 0. Combining all this,

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \pi \]

as desired. 

[02.10] Compute \( \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} \)

**Discussion:** As in the previous example, the infinite integral is a limit of finite limits

\[ \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \lim_{R \rightarrow +\infty} \int_{-R}^{R} \frac{dx}{x^4 + 1} \]

The denominator has zeros at eighth roots of unity, namely, \( \zeta = \zeta_8 = e^{\pi i/4}, \zeta^3 = e^{3\pi i/4}, \zeta^5 = e^{5\pi i/4}, \zeta^7 = e^{7\pi i/4}. \) Let \( \gamma_R \) be the path from \(-R\) to \(R\) along the real line, and then along the auxiliary arc, the circle of radius \( R \) in the upper half-plane, from \(+R\) back to \(-R\). The integral over the arc is estimated via the trivial estimate:

\[ \left| \int_{\text{arc } R} \frac{dz}{x^4 + 1} \right| \leq \text{length}(\text{arc } R) \cdot \sup_{\text{on arc } R} \left| \frac{1}{z^4 + 1} \right| \leq \pi R \cdot \frac{1}{(R-1)^4} \]

This goes to 0 as \( R \rightarrow +\infty. \) Thus, using the Residue Theorem, the original integral is

\[ \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \lim_{R \rightarrow +\infty} \int_{\gamma_R} \frac{dz}{1+z^4} = \lim_{R \rightarrow +\infty} 2\pi i \operatorname{Res}_{z=\zeta, \zeta^3} \frac{1}{1+z^4} \]

\[ = 2\pi i \left( \frac{1}{(\zeta - \zeta^3)(\zeta - \zeta^5)(\zeta - \zeta^7)} + \frac{1}{(\zeta^3 - \zeta)(\zeta^3 - \zeta^5)(\zeta^3 - \zeta^7)} \right) \]

recalling the convenient fact that the residue at \( z_o \) of \( g(z)/(z-z_o) \) for \( g \) holomorphic at \( z_o \) is \( g(z_o). \) This is

\[ 2\pi i \left( \frac{1}{(\sqrt{2})(2\zeta)(i\sqrt{2})} + \frac{1}{(-\sqrt{2})(i\sqrt{2})(2i\zeta)} \right) = \frac{\pi i}{2} \left( \frac{1}{\zeta} + \frac{1}{\zeta^3} \right) = \frac{\pi \zeta^2}{2} \cdot \frac{1-i}{\zeta} = \frac{\pi}{2} \cdot \frac{1+i}{\sqrt{2}} \cdot (1-i) = \frac{\pi}{\sqrt{2}} \]

as required.

[02.11] Compute \( \int_{-\infty}^{\infty} \frac{x \, dx}{x^2 + 1} \)

**Discussion:** This example illustrates the relevance of symmetry. That is, the function \( 1/(x^4 + 1) \) is even, and the function \( x \) is odd, so \( x/(x^4 + 1) \) is odd. But then the integral over the whole real line is equal to its own negative, under the change of variables \( x \rightarrow -x, \) so is 0.

[02.12] Compute \( \int_{-\infty}^{\infty} \frac{x^2 \, dx}{x^4 + 1} \)

**Discussion:** By residues, using a semi-circle in the upper (or lower) half-plane as auxiliary arc. Picks up residues at the two primitive 8\text{th} roots of unity in the upper (or lower) half-plane...
[02.13] Compute \( \int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 + 1} \, dx \) with real \( t \).

**Discussion:** By residues, depending on the sign of \( t \in \mathbb{R} \), using a semi-circle in the upper/lower half-plane. The choice depends on the sign of \( t \), so that \( e^{itz} \) is bounded in \( z \) in the corresponding half-plane.

As usual, the integral is equal to a contour/path integral, whose sense we emphasize by using \( z \) as a dummy variable instead of \( x \).

As usual, the infinite integral is a limit of integrals \( \int_{-R}^{R} \) as \( R \to +\infty \). For \( t \geq 0 \), \( z \to e^{itz} \) is bounded in \( \text{Im}(z) \geq 0 \), because for real \( t \) and \( z = x + iy \),

\[
|e^{itz}| = e^{\text{Re}(itz)} = e^{-ty}
\]

Thus, the trivial estimate of the integral on the auxiliary arc \( \alpha_R \) (the half-circle in the upper half-plane) is that it’s dominated by

\[
\text{length } \alpha_R \times \text{max of integrand on arc} \leq \pi R \times \frac{1}{(R-1)^2} \to 0 \quad \text{(as } R \to +\infty)\]

Thus, for \( t \geq 0 \), by residues, the integral is essentially the sum of residues of the integrand in the upper half-plane: it is

\[
2\pi i \cdot \text{Res}_{z=i} \frac{e^{itz}}{z^2 + 1} = 2\pi i \cdot \frac{e^{it(i)}}{i + i} = \pi \cdot e^{-t}
\]

using the general fact that \( \text{Res}_{z=z_o} f(z)/(z - z_o) = f(z_o) \) for \( f \) holomorphic at \( z_o \).

For \( t \leq 0 \), the auxiliary arc is a semi-circle in the lower half-plane. Because the closed path is traced in a mathematically negative direction, the **negative** of \( 2\pi i \) times the residues is picked up, giving

\[
-2\pi i \text{Res}_{z=-i} \frac{e^{itz}}{z^2 + 1} = -2\pi i \cdot \frac{e^{i(-i)}}{(-i) - i} = \pi \cdot e^t
\]

We can express both simultaneously as \( \pi e^{-|t|} \). ///

[02.14] Compute \( \int_{-\infty}^{\infty} \frac{\sin(tx)}{x^2 + 1} \, dx \) with real \( t \).

**Discussion:** Another short-circuited issue: the integrand is **odd**, so its integral over the whole real line (stable under \( x \to -x \)) is 0, by pure thought. ///

[02.15] Compute \( \int_{-\infty}^{\infty} \frac{\cos(tx)}{x^2 + 1} \, dx \) with real \( t \).

**Discussion:** Use \( \cos(tx) = \frac{e^{itx} + e^{-itx}}{2} \), and for auxiliary arcs use a semi-circle in the upper half-plane for one term, and in the lower half-plane for the other, also depending on the sign of \( t \). This reduces to a previous example. In fact, since \( 1/(x^2 + 1) \) is **even**, and \( \cos(tx) \) is the average of \( e^{\pm itx} \), the outcome is the same as with \( e^{itx} \) in place of \( \cos(tx) \), namely, \( \pi e^{-|t|} \). ///

[02.16] Compute \( \int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2} \, dx \) with real \( t \).

**Discussion:** As usual, to compute by residues, we closed up the paths \([-R,R]\) by auxiliary arcs. As in other example, the choice of auxiliary arc depends on the sign of \( t \in \mathbb{R} \): for \( t \geq 0 \), \( z \to e^{itz} \) is bounded in the **upper** half-plane, so we use an upper half-plane semi-circle (at 0) of radius \( R \), while for \( t \leq 0 \), the exponential
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is bounded in the lower half-plane, and we use a lower half-plane semi-circle of radius \( R \), traversing a closed curve in the negative direction.

For \( t \geq 0 \), the trivial/easy estimate on the \( R^{th} \) auxiliary arc is that it is dominated by

\[
\text{length} \cdot \text{max} \text{imum absolute value of integrand} \leq \pi R \times \frac{1}{(R-1)^2} \rightarrow 0 \quad \text{(as } R \rightarrow +\infty) \]

Thus, in this case, for \( R \geq 2 \), the integral can be evaluated by residues, but/and there are no singularities of \( e^{itz}/(z^2 + 1) \) in the upper half-plane, so producing 0.

For \( t \leq 0 \), the trivial/easy estimate on the \( R^{th} \) auxiliary arc again shows that it goes to 0, so the integral can be evaluated by residues. The point \( z = -i \) is the only singularity (in the lower half-plane), and the integral is (integrating in the negative direction)

\[
-2\pi i \times \text{Res}_{z=-i} \left( \frac{e^{itz}}{(z-(-i))^2} \right) = -2\pi i \times \frac{d}{dz} e^{itz} \bigg|_{z=-i} 
\]

by the somewhat general formula

\[
\text{Res}_{z=z_0} \left( \frac{f(z)}{(z-z_0)^n} \right) = \frac{f^{(n-1)}(z_0)}{(n-1)!} 
\]

This gives

\[
-2\pi i \cdot (it) \cdot e^{it(-i)} = 2\pi t \cdot e^t
\]

Thus, in a form that perhaps emphasizes the qualitative features, the integral is

\[
\left\{ \begin{array}{ll}
0 & \text{(for } t \geq 0) \\
2\pi |t| \cdot e^{-|t|} & \text{(for } t \leq 0) 
\end{array} \right.
\]

[02.17] Compute \( \int_{-\infty}^{\infty} e^{itx} dx \) with real \( t \).

Discussion: The same argument as in the previous example gives 0 for \( t \geq 0 \) and for \( t \leq 0 \) it is

\[
-2\pi i \cdot (it)^{n-1} \cdot (n-1)! \cdot e^{it(-i)} = -2\pi \cdot i^n \cdot t^{n-1} \cdot (n-1)! \cdot e^{-|t|}
\]

as required.

[02.18] Compute \( \int_{0}^{\infty} \frac{x \, dx}{1 + x^3} \)

Discussion: As usual, the integral is the limit of finite integrals \( \int_{0}^{R} \) as \( R \rightarrow +\infty \). Let \( \gamma_R \) be the path from 0 to \( R \) along the real line, then counter-clockwise along the circle of radius \( R \) to \( R \cdot e^{2\pi i/3} \), then back along the straight line to 0. This path is chosen because the integral from \( R \cdot e^{2\pi i/3} \) to 0 is very simply related to the original:

\[
\int_{0}^{R} \frac{e^{2\pi i/3} t \, d(e^{2\pi i/3} t)}{1 + (e^{2\pi i/3} t)^3} = -e^{4\pi i/3} \int_{0}^{R} \frac{t \, dt}{1 + t^3}
\]

The integral along the arc is easily estimate by the trivial estimate:

\[
\left| \int_{\text{arc } R} \frac{z \, dz}{1 + z^3} \right| \leq \text{length(} \text{arc } R \text{)} \cdot \sup_{\text{on } \text{arc } R} \left| \frac{z}{1 + z^3} \right| \leq \frac{2\pi R}{3} \cdot \frac{R}{(R-1)^3}
\]
which goes to 0 as \( R \to +\infty \). The integral over \( \gamma_R \) can be evaluated by residues: for \( R > 1 \), there is a single singularity inside \( \gamma_R \), at the sixth root of unity \( \zeta = \zeta_6 = e^{\pi i/3} \). Noting that

\[ z^3 + 1 = (z + 1)(z^2 - z + 1) = (z + 1)(z - \zeta)(z - \zeta^{-1}) \]

and that \(-e^{4\pi i/3} = \zeta\), we have

\[
(1 + \zeta) \int_0^\infty \frac{x \, dx}{1 + x^3} = \lim_{R \to \infty} \int_{\gamma_R} \frac{z \, dz}{1 + z^3} = 2\pi i \text{Res}_{z=\zeta} \frac{z}{1 + z^3} = 2\pi i \frac{\zeta}{(\zeta + 1)(\zeta - \zeta^{-1})}
\]

so

\[
\int_0^\infty \frac{x \, dx}{1 + x^3} = 2\pi i \frac{\zeta}{(\zeta + 1)^2(\zeta - \zeta^{-1})} = 2\pi i \frac{1}{(\zeta + 1)(\zeta^{-1} + 1)(i\sqrt{3})} = \frac{2\pi}{(\frac{3}{4} + \frac{1}{4})\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}
\]

as required. ///

[02.19] Compute \( \int_0^\infty \frac{x^{1/4} \, dx}{1 + x^2} \)

**Discussion:** We can replace \( x \) by \( x^4 \), and reduce to some previous examples, or look at this as a special case of

\[
\int_0^\infty \frac{x^s \, dx}{1 + x^2} \quad \text{(with } -1 < \text{Re}(s) < 1)\]

The latter is amenable to use of the keyhole or Hankel contour... [... iou ...]

///

[02.20] Compute \( \frac{1}{1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots} \)

**Discussion:** Arrange to evaluate the infinite sum by residues, by using the function \( 2\pi i / (e^{2\pi i z} - 1) \), which we will check has simple poles with residues 1 at integers, and for \( z \) bounded away from integers is bounded. Granting that for a moment, letting \( \gamma_T \) be a counter-clockwise path around the square with vertices \( \pm T \pm iT \) with \( T \in \frac{1}{2} + \mathbb{Z} \), by residues

\[
\int_{\gamma_T} \frac{2\pi i}{e^{2\pi i z} - 1} \cdot \frac{1}{z^4} \, dz = \sum_{0 \leq |n| < T} \text{Res}_{z=n} \frac{2\pi i}{e^{2\pi i z} - 1} \cdot \frac{1}{z^4} = \sum_{0 < |n| < T} \frac{1}{n^4} + \text{Res}_{z=0} \frac{2\pi i}{e^{2\pi i z} - 1} \cdot \frac{1}{z^4}
\]

Because of the division by \( z^4 \), the latter residue is visibly the coefficient of \( z^3 \) in the Laurent expansion of \( 2\pi i / (e^{2\pi i z} - 1) \), which is determined by expanding \( 1 / (e^z - 1) \)

\[
\frac{1}{e^z - 1} = \frac{1}{(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots) - 1} = \frac{1}{z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots}
\]

\[
= \frac{1}{z} \cdot \frac{1}{\frac{1}{z} + \left( \frac{1}{2} + \frac{z^2}{6} + \ldots \right)} = \frac{1}{z} \left( 1 - \left( \frac{z}{2} + \frac{z^2}{6} + \ldots \right) + \left( \frac{z}{2} + \frac{z^2}{6} + \ldots \right)^2 - \left( \frac{z}{2} + \frac{z^2}{6} + \ldots \right)^3 + \ldots \right)
\]

The \( z^3 \) coefficient of the latter is

\[
-\frac{1}{5!} + 2 \cdot \left( \frac{1}{2} \cdot \frac{1}{4!} + 1 \cdot \left( \frac{1}{3!} \right)^2 \right) - 3 \cdot \left( \frac{1}{2!} \right)^2 \cdot \frac{1}{3!} + \left( \frac{1}{2!} \right)^4 = -\frac{1}{120} + \frac{1}{24} + \frac{1}{36} - \frac{1}{8} + \frac{1}{16}
\]

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Replacing \( z \) by \( 2\pi iz \) in that Laurent expansion, and multiplying from the \( 2\pi i \) from the numerator multiplies this by \( (2\pi i)^4 = 16\pi^4 \), giving \( \pi^4 \) times

\[
- \frac{16}{120} + \frac{16}{24} + \frac{16}{36} - \frac{16}{8} + \frac{16}{16} = - \frac{2}{3} + \frac{2}{3} + 4 - 1 = - \frac{6 + 30 + 20 - 45}{45} = - \frac{1}{45}
\]

so \(-\pi^4/45 \). Thus, still granting that everything works out, we have

\[
\int_{\gamma_T} \frac{2\pi i}{e^{2\pi iz}} \frac{1}{z^4} \, dz = \sum_{0 < |n| < T} \frac{1}{n^4} + \text{Res}_{z=0} \frac{2\pi i}{e^{2\pi iz} - 1} \cdot \frac{1}{z^4} = \sum_{0 < |n| < T} \frac{1}{n^4} - \frac{\pi^4}{45}
\]

Taking the limit, the integral goes to 0, so

\[
0 = \lim_{T} \sum_{0 < |n| < T} \frac{1}{n^4} - \frac{\pi^4}{45} = 2 \cdot \sum_{n \geq 1} \frac{1}{n^4} - \frac{\pi^4}{45}
\]

giving the claimed result. For the other details:

The function \( 2\pi i/(\exp(2\pi iz) - 1) \) has no singularities unless the denominator is 0, which occurs exactly at integers. It is \( \mathbb{Z} \)-periodic, so to check that its residue at 0 is 1: as above,

\[
\frac{2\pi i}{
\exp(2\pi iz) - 1
} = \frac{2\pi i}{
(1 + (2\pi iz) + (\frac{(2\pi iz)^2}{2}) + \ldots) - 1
} = \frac{2\pi i}{
(2\pi iz + (\frac{(2\pi iz)^2}{2}) + \ldots
} = \frac{1}{z + \frac{(2\pi iz)^2}{2} + \ldots
} = \frac{1}{z}
\]

To check that this function is bounded for \( z \) away from integers, first observe that \( |\exp(2\pi iz)| \leq \frac{1}{e} \) for \( \text{Im}(z) \geq \frac{1}{2\pi} \), and \( |\exp(2\pi iz)| \geq e \) for \( \text{Im}(z) \leq -\frac{1}{\pi} \). In both cases, \( \exp(2\pi iz) - 1 \) is bounded away from zero, so \( 2\pi i/(\exp(2\pi iz) - 1) \) is bounded.

For \( |\text{Im}(z)| \leq \frac{1}{2\pi} \), again use periodicity, to reduce to the set where \( |\text{Im}(z)| \leq \frac{1}{2\pi}, 0 \leq \text{Re}(z) \leq 1 \), and \( |z - 0| \geq \frac{1}{2} \) and \( |z - 1| \geq 1 \). This set is compact, and \( 2\pi i/(\exp(2\pi iz) - 1) \) is continuous on it, so is bounded. This completes the checking of the background details to make things work.

[02.21] Show that \( z^n + z - 1 \) has \( n \) zeros inside the circle \( |z| = 2 \).

**Discussion:** Implicitly, probably \( n \geq 3 \) or something of this sort. This is about Rouché’s theorem, or possibly a simpler argument, since the question only involves polynomials. An orthodox approach says that \( f(z) = z^n - 1 \) and \( g(z) = z^n + z - 1 \) have the same number of roots inside \( |z| = 2 \) if \( |f(z) - g(z)| \) \(< \|f(z)\| \) on that circle. As designed,

\[
|f(z) - g(z)| = |z|^2 - 2^n - 1 \leq |z^n - 1| = |f(z)| \quad \text{(on } |z| = 2 \text{)}
\]

so that Rouché’s theorem is applicable.

[02.22] Are there complex zeros of \( z - \cos z \) beyond the obvious one on \( \mathbb{R} \)?

**Discussion:** This is about Rouché’s theorem, but not so easy to definitively address.

[02.23] For a bounded sequence of complex numbers \( c_n \), prove that \( \sum_{n=0}^{\infty} c_n \frac{z^n}{z^n + 1} \) converges to a holomorphic function on \( |z| < 1 \).
Discussion: Each summand is holomorphic on $|z| < 1$, because of the quotient rule, and that the numerator and denominator are polynomials, hence holomorphic.

To prove that the sum $\sum_n f_n$ of a sequence of holomorphic functions on $|z| < 1$ is itself holomorphic, it suffices to prove that the convergence is uniform on compacts. The compact subsets of the open disk are all contained in compact disks $|z| \leq r$ for $r < 1$, so it suffices to consider just those sets $|z| \leq r$.

Given $r < 1$, there is large-enough $N$ such that $r^n \leq \frac{1}{2}$ for all $n \geq N$, for example taking $N \geq \frac{\log \frac{1}{2}}{\log r}$. For $|z| \leq r$ and $n \geq N$,

$$\left| \frac{z^n}{1 + z^n} \right| \leq \left| \frac{|z|^n}{1 - \frac{1}{2}} \right| \leq 2r^n$$

Thus, given $0 < r < 1$, let $N$ so that $r^n \leq \frac{1}{2}$ for all $n \geq N$. Given $\varepsilon > 0$, for $m, n \geq N$, with $|c_n| \leq B$ for all $n$,

$$\left| \sum_{m \leq j < n} c_n \frac{z^j}{1 + z^n} \right| \leq B \cdot \sum_{m \leq j < n} 2r^j < B \cdot \sum_{m \leq j < \infty} 2r^j \leq B \frac{2r^m}{1 - r}$$

Increasing $N$ if necessary, this is smaller than $\varepsilon$.

There are several viable variant approaches. Among others: expanding the power series for each $z^n/((z^n + 1)$, although one should be careful not to suggest that a sum of holomorphic functions on a disk is holomorphic on that disk, since $\sum_n c_n z^n$ can have arbitrary radius of convergence, while the summands $c_n z^n$ have infinite radius of convergence. Invocation of Morera’s theorem also works here.

[02.24] Let $f$ be a continuous, bounded real-valued function on $\mathbb{R}$, extending to a bounded, holomorphic function on the upper half-plane $\mathbb{D}$. Show $f$ is constant.

Discussion: This is a combination of the Reflection Principle and Liouville’s Theorem. Namely, by the Reflection Principle, defining $f$ in the lower half-plane by $f(z) = \overline{f(\overline{z})}$ gives a holomorphic function on $\mathbb{C}$. This extension is still bounded, so by Liouville is constant.

[02.25] Evaluate the Fourier transform $\int_{-\infty}^{\infty} e^{-itz} \cdot \frac{1}{(x + i)^s} \, dx$ for complex $s$ with $\text{Re}(s) > 1$, using the Gamma function.

Discussion: My preferred approach to this, while not the shortest, illustrates some important methodological and technical points.

Recall that the Identity Principle gives

$$\int_{0}^{\infty} e^{-uz} u^s \frac{du}{u} = z^{-s} \Gamma(s) \quad \text{(for Re}(z) > 0 \text{ and Re}(s) > 0)$$

Using this identity in the problem at hand,

$$\int_{-\infty}^{\infty} e^{-itz} \frac{1}{(x + i)^s} \, dx = i^{-s} \int_{-\infty}^{\infty} e^{-itz} \frac{1}{(1 - ix)^s} \, dx = i^{-s} \frac{1}{\Gamma(s)} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-itz} e^{-u(1-it)x} u^s \frac{du}{u} \, dx$$

Changing the order of integration, if justifiable, would give

$$i^{-s} \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{-u} \left( \int_{-\infty}^{\infty} e^{i(u-t)x} \, dx \right) u^s \frac{du}{u}$$

The difficulty is that the inner integral is not at all convergent in a classical, pointwise sense. Thus, with hindsight, the interchange of integrals is not justifiable in classical terms.
Nevertheless, that integral should remind us of Fourier Inversion: for nice-enough functions,
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \left( \int_{-\infty}^{\infty} e^{-ixu} f(u) \, du \right) d\xi \]
In particular, there is an illuminating heuristic, or near-proof, for Fourier Inversion, involving the same not-classically-justifiable interchange of integrals:
\[ \int_{-\infty}^{\infty} e^{ix\xi} \left( \int_{-\infty}^{\infty} e^{-i\xi u} f(u) \, du \right) d\xi = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{i\xi(x-u)} \, d\xi \right) f(u) \, du \]  
Since we know that this should be \( 2\pi \cdot f(x) \), it must be that, in effect,
\[ \int_{-\infty}^{\infty} e^{i\xi(x-u)} \, d\xi = 2\pi \cdot \delta(x-u) \quad \text{(Dirac delta)} \]
Granting this heuristic for a moment, the integral at hand would become
\[ 2\pi \cdot i^{-s} \frac{1}{\Gamma(s)} \int_{0}^{\infty} e^{-u} \delta(u-t) \frac{u^{s}}{u} \, du = \begin{cases} \frac{2\pi}{i^{s} \Gamma(s)} e^{-t^{s-1}} & \text{(for } t \geq 0) \\ 0 & \text{(for } t < 0) \end{cases} \]
In our context this is only a heuristic, but it suggests the correct value for the integral, and we can attempt to check the outcome of the heuristic, via Fourier Inversion. Thus, disregarding the constant \( 2\pi/i^{s}\Gamma(s) \) for a moment, compute the inverse Fourier transform of
\[ F(t) = \begin{cases} e^{-t^{s-1}} & \text{(for } t \geq 0) \\ 0 & \text{(for } t < 0) \end{cases} \]
This is
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} F(\xi) \, d\xi = \frac{1}{2\pi} \int_{0}^{\infty} e^{ix\xi} e^{-\xi \xi^{s-1}} \, d\xi = \frac{1}{2\pi} \int_{0}^{\infty} e^{ix\xi} e^{-\xi} \xi^{s} \frac{d\xi}{\xi} = \frac{1}{2\pi} \int_{0}^{\infty} e^{-\xi(1-ix)} \xi^{s} \frac{d\xi}{\xi} = \frac{1}{2\pi} \frac{1}{(1-ix)^{s}} \Gamma(s) = \frac{1}{2\pi} i^{s} \frac{1}{(x+i)^{s}} \Gamma(s) \]
by the same identity. Thus, all the constants correctly cancel, and by Fourier Inversion we see that the heuristic gave the true outcome:
\[ \int_{-\infty}^{\infty} e^{-itx} \frac{1}{(x+i)^{s}} \, dx = \begin{cases} \frac{2\pi}{i^{s} \Gamma(s)} e^{-t^{s-1}} & \text{(for } t \geq 0) \\ 0 & \text{(for } t < 0) \end{cases} \]
as desired.  

\[ 02.26 \] Show that \( f(z) = \int_{0}^{1} \frac{dt}{t \cdot z + (1-t) \cdot z_{0}} \) is holomorphic at any \( z_{1} \) such that 0 is not on the straight line segment with endpoints \( z_{0} \) and \( z_{1} \). Find the radius of convergence of its power series expanded at \( z_{0} = -4 + 3i \).

**Discussion:** As with the case \( z_{0} = 1 \), holomorphy is proven via Morera’s Theorem, for example.

For any \( z_{0} \) such that the line segment connecting \( z_{0} \) and \( -4 + 3i \) does not pass through 0, the corresponding function is holomorphic at \( -4 + 3i \), so admits a power series expansion there. From Cauchy theory, this
power series will converge on the largest open disk centered at $-4 + 3i$ on which there is a holomorphic function agreeing with $f(z)$.

Because of the potential blow-up of the integral, not to mention knowing that $\log 0$ cannot have a value making the function holomorphic, no one of these functions $f(z)$ can be holomorphic at 0, so 0 is not contained in any disk on which $f(z)$ is holomorphic. We show that there is no other obstacle.

The functions $f(z)$ defined via different $z_o$ only differ by constants, the value of the integral of $1/w$ from one $z_o$ to another. Thus, in particular, we could consider $z_o = -4 + 3i$ without loss of generality, in the sense that if we find radius of convergence equal to the distance to 0 (namely, 5), then, since we cannot do any better, we’re done.

The function $f(z)$ defined with $z_o = -4 + 3i$ is holomorphic on the slit plane obtained by removing from $\mathbb{C}$ the ray from 0 passing through $-(-4 + 3i)$. The largest disk centered at $-4 + 3i$ in this half-plane indeed has radius 5, the distance from $-4 + 3i$ to 0.  

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