Complex analysis examples discussion 03

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2020-21/cx_discussion_03.pdf]

[03.1] Adapt the reflection principle to show that a holomorphic function on the unit disk, extending to a continuous function on the closed unit disk, with \( |f(z)| = 1 \) on the unit circle, extends to a holomorphic function on \( \mathbb{C} \) except for finitely-many poles. (Hint: for example, \( z \to \frac{z-i}{z+i} \) maps the real line to the unit circle.)

Discussion: The inverse Cayley map \( C^{-1}(z) = \frac{z-i}{z+i} \) maps the upper half-plane to the disk, and the real line to the unit circle. Thus, \( F = C^{-1} \circ f \circ C \) is holomorphic on the upper half-plane taking values in the upper half-plane, extending to a real-valued continuous function on \( \mathbb{R} \), satisfies the hypotheses of the most standard reflection principle. Thus, by the reflection principle, \( F \) extends by

\[
F(\overline{z}) = \overline{F(z)} \quad \text{(for } z \in \mathbb{R})
\]

Let \( \alpha \) be the complex conjugation map. The Cayley map interacts in a coherent way with conjugation: \( C \circ \alpha = \alpha \circ C^{-1} \), and \( C^{-1} \circ \alpha = \alpha \circ C^{-1} \).

Letting \( F \) still denote the extension,

\[
f = C^{-1} \circ F \circ C = C^{-1} \circ (\alpha \circ F \circ \alpha) \circ C
\]

by using the formula for the extension of \( F \) to the lower half-plane. Now rearrange to rewrite the latter expression in terms of \( f \) itself. First, the interaction of \( \alpha \) and \( C \) gives

\[
\alpha \circ C \circ F \circ C^{-1} \circ \alpha = \alpha \circ C^2 \circ (C^{-1} \circ F \circ C) \circ C^{-2} \circ \alpha = \alpha \circ C^2 \circ f \circ C^{-2} \circ \alpha
\]

The Cayley map has the important property that \( C^2(z) = C^{-2}(z) = 1/z \), so this gives

\[
f(z) = \alpha \circ \frac{1}{f(1/\overline{z})} = 1/f(1/\overline{z})
\]

This is the desired reflection formula for the circle. Note that zeros of \( f \) give rise to poles of the reflected part. The hypotheses on \( f \) in the disk assure that it has finitely-many zeros there, so the reflected part has finitely-many poles. ///

[03.2] Show that \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \).

Discussion: First, a change of variables

\[
\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t^2} \frac{dt}{t} = 2 \int_0^\infty e^{-t^2} t \frac{dt}{t} = \int_\mathbb{R} e^{-t^2} dt
\]

and then the standard calculus device: squaring and converting to polar coordinates:

\[
\left( \int_\mathbb{R} e^{-t^2} dt \right)^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta = 2\pi \int_0^\infty e^{-r^2} r dr
\]

\[
= \pi \int_0^\infty e^{-r^2} 2r dr = \pi \int_0^\infty e^{-r} dr = \pi
\]

Thus, \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \). ///
Alternatively, the functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ gives

$$\Gamma\left(\frac{1}{2}\right)^2 = \Gamma\left(\frac{1}{2}\right) \cdot \Gamma\left(1 - \frac{1}{2}\right) = \frac{\pi}{\sin \frac{\pi}{2}} = \pi$$

and we reach the same conclusion.

[03.3] Show that $\Gamma(s) = \Gamma(s)$ for all $s \in \mathbb{C}$.

**Discussion:** This is an instance of application of the Identity Principle. Namely, from its integral representation, $\Gamma(s)$ is $\mathbb{R}$-valued for $s \in (0, +\infty)$. Thus, for $s \in (0, +\infty)$, $\Gamma(s) = \Gamma(\overline{s})$. Thus, $s \rightarrow \Gamma(s)$ is holomorphic. By the Identity Principle, we have the asserted equality everywhere.

We might want to recall the computation that for $f$ holomorphic $z \rightarrow f(z)$ is again holomorphic, by checking complex differentiability:

$$\frac{f(z + h) - f(z)}{h} = \frac{f(\overline{z + h}) - f(\overline{z})}{\overline{h}}$$

Since $h \rightarrow 0$ is equivalent to $\overline{h} \rightarrow 0$, the limit as $\overline{h} \rightarrow 0$ of the expression under the complex conjugation on the right-hand side is $f'(z)$. In particular, the limit exists. Thus, $z \rightarrow f(z)$ is holomorphic.

[03.4] Show that $|\Gamma\left(\frac{1}{2} + it\right)| = \sqrt{\frac{\pi}{\cosh \pi t}}$ for real $t$. (Thus, in contrast to the horizontal super-exponential growth of $n \rightarrow n!$, the vertical behavior is exponential decrease.)

**Discussion:** Use the functional equation and $\Gamma(s) = \Gamma(\overline{s})$:

$$|\Gamma\left(\frac{1}{2} + it\right)|^2 = \Gamma\left(\frac{1}{2} + it\right) \cdot \Gamma\left(\frac{1}{2} + it\right) = \Gamma\left(\frac{1}{2} + it\right) \cdot \Gamma\left(1 - \left(\frac{1}{2} + it\right)\right) = \frac{2\pi i}{\sin \pi\left(\frac{1}{2} + it\right)} = \frac{2\pi i}{e^{\pi i (1+it)} - e^{-\pi i (1+it)}} = \frac{2\pi i}{ie^{-\pi t} + ie^{\pi t}} = \frac{\pi}{\cosh \pi t}$$

Thus,

$$|\Gamma\left(\frac{1}{2} + it\right)| = \sqrt{\frac{\pi}{\cosh \pi t}}$$

as claimed.

[03.5] Prove that $f(z) = \int_0^1 \frac{e^{tz}}{t^2 + 1} \, dt$ is holomorphic.

**Discussion:** The simplest argument might be to invoke Morera’s theorem after changing order of integration. The change of order is easily justifiable, since one is looking at a continuous function of two variables. That is, for each $t \in [0, 1]$, the function $z \rightarrow \frac{e^{tz}}{t^2 + 1}$ is holomorphic, and the function of two variables is continuous. Thus, letting $\gamma$ be a small triangle,

$$\int_0^1 \int_{\gamma} \frac{e^{tz}}{t^2 + 1} \, dz \, dt = \int_0^1 \int_{\gamma} \frac{e^{tz}}{t^2 + 1} \, dz \, dt = \int_0^1 0 \, dt = 0$$

by applying Cauchy’s theorem to $z \rightarrow \frac{e^{tz}}{t^2 + 1}$. By Morera, $f(z)$ is continuous.

Another approach is to view the integral as a uniform limit of a sequence of finite (Riemann) sums, each of which is holomorphic, being a finite sum of holomorphic functions, and then invoke the holomorphy of uniform (on compacts) limits of holomorphic functions.
[03.6] Prove that \( f(z) = \int_0^{\infty} \frac{e^{-tz}}{t^2 + 1} \) is holomorphic for \( \text{Re}(z) > 0 \).

Discussion: Using the previous example, it would suffice to show that the sequence of finite integrals
\[
f_n(z) = \int_0^{n} \frac{e^{-tz}}{t^2 + 1}
\]
converges uniformly to \( f(z) \) for \( z \) in compact subsets of \( \text{Re}(z) > 0 \), since these finite integrals are holomorphic functions, via Morera.

For fixed \( \delta > 0 \) and \( \text{Re}(z) \geq \delta \), for \( N \leq m \leq n \),
\[
\left| f_m(z) - f_n(z) \right| \leq \int_m^n \frac{e^{-t\delta}}{t^2 + 1} \leq \int_N^{\infty} e^{-t\delta} dt = \frac{e^{-N\delta}}{\delta}
\]
This can be made smaller than a given \( \delta > 0 \) by taking \( N \) sufficiently large. ///

[03.7] Compute \( \int_0^{\infty} \frac{x^s}{1+x^2} \) dx.

Discussion: The integral is absolutely convergent for \(-1 < \text{Re}(s) < 1\). Implicitly,
\[
x^s = e^{s \log x}
\]
where the logarithm is the one which is real-valued on \((0, +\infty)\). Use the Hankel/keyhole contour. First, the integral itself is a limit
\[
\int_0^{\infty} \frac{x^s}{1+x^2} \, dx = \lim_{\epsilon \to 0^+ \epsilon < R} \int_{\epsilon}^{R} \frac{x^s}{1+x^2} \, dx.
\]
Let \( H_{\epsilon,R} \) be the Hankel/keyhole contour that comes from \( R \) along the real line to \( \epsilon \), then traces a circle of radius \( \epsilon \) around 0 clockwise to \( \epsilon \), then back out to \( R \). Let \( H_\epsilon \) be the limiting case as \( R \to +\infty \). We want the integral along that last part of the path, the outbound part from \( \epsilon \) back out to \( R \), to be the original integral \( \int_\epsilon^{R} \frac{x^s}{(x^2 + 1)} \, dx \). That is, we want the value of \( x^s \) to match.

On that small circle, the argument of \( x \) changes continuously, with a net decrease of \( 2\pi \) from its value on the in-bound part of the path. Requiring that \( x^s \) change continuously on that small circle, and be \( e^{s \log x} \) with real-valued \( \log x \) after traversing \( 2\pi \) radians counter-clockwise, requires that \( x^s \) be \( e^{s(\log x + 2\pi i)} \) on the in-bound path. Thus,
\[
\int_{\text{outbound+inbound}} \frac{x^s \, dx}{1+x^2} = \left(1 - e^{2\pi is}\right) \int_{\epsilon}^{R} \frac{x^s}{1+x^2} \, dx;
\]
Further, the main point of the keyhole trick is that, surprisingly, the limit over \( \epsilon \to 0^+ \) is reached in finite time, in the sense that there is sufficiently small \( \epsilon_0 > 0 \) such that
\[
\lim_{\epsilon \to 0^+} \int_{H_{\epsilon,R}} \frac{x^s \, dx}{1+x^2} = \int_{H_{\epsilon_1,R}} \frac{x^s \, dx}{1+x^2} \quad \text{(for all positive } \epsilon_1 < \epsilon_0)\]
Recall the proof: for \( 0 < \epsilon_1 < \epsilon_0 \), let \( \gamma_{\epsilon_0, \epsilon_1} \) be the closed path that traces counter-clockwise around the circle of radius \( \epsilon_0 \) from \( \epsilon_0 \) back to \( \epsilon_0 \), then left to \( \epsilon_1 \), then clockwise around a circle of radius \( \epsilon_1 \) back to \( \epsilon_1 \), then right to \( \epsilon_0 \). In the interior of this path, the integrand is holomorphic. Adding the integral over \( \gamma_{\epsilon_0, \epsilon_1} \) to the integral over \( H_{\epsilon_1,R} \) makes the integrals from \( \epsilon_0 \) to \( \epsilon_1 \) (inbound) and from \( \epsilon_1 \) to \( \epsilon_0 \) (outbound) cancel, and the integrals around the circles of radius \( \epsilon_1 \) cancel, leaving \( H_{\epsilon_0,R} \). (Yes, one should draw a picture.) To evaluate
\[
\int_{H_{\epsilon_1,R}} \frac{x^s \, dx}{1+x^2}
\]
Thus, the ends of the box are easily estimated: since 

\[ \int \]

Then the arg \( x \) poles, at \( x \) times the sum of residues in its interior. Inside that path, for small \( \varepsilon \) and large \( R \), there are exactly two poles, at \( x = \pm i \), and both are simple. The value of arg \( x \) at \( i \) is obtained by moving counter-clockwise from the arg \( x = 0 \) on \((0, +\infty)\), giving \( \pi \). The argument at \(-i\) is obtained by continuing counter-clockwise, giving \( \frac{3\pi}{2} \). Thus,

\[
\text{sum of residues} = \frac{e^{\pi i} s}{(-i) - i} + \frac{e^{3\pi i} s}{i - (-i)} = \frac{e^{\pi i} s}{-2i} + \frac{e^{3\pi i} s}{2i}
\]

In summary,

\[
\int_0^\infty \frac{x^s}{1 + x^2} \, dx = \frac{1}{1 - e^{2\pi i s}} \lim_R \int_{H_{1+R}} \frac{x^s}{1 + x^2} \, dx = \frac{2\pi i}{1 - e^{2\pi i s}} \left( \frac{e^{\pi i} s}{-2i} + \frac{e^{3\pi i} s}{2i} \right)
\]

\[
= \frac{\pi}{e^{2\pi i s} - 1} \left( e^{\pi i s} - e^{3\pi i s} \right) = \pi \frac{e^{\pi i s} - e^{-\pi i s}}{2} = \frac{\pi}{2 \cos \frac{\pi s}{2}}
\]

[03.8] Compute \( \int_{-\infty}^\infty e^{-ix} e^{-x^2} \, dx \)

**Discussion:** The exponentials can be combined, and then complete the square:

\[-\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-ix} e^{-x^2} \, dx = \int_{-\infty}^{\infty} e^{-ix} e^{-x^2} \, dx = \int_{-\infty}^{\infty} e^{-(x^2+ix^2)} \, dx
\]

\[
= \int_{-\infty}^{\infty} e^{-(x^2+ix+\frac{\xi^2}{4})-rac{\xi^2}{4}} \, dx = e^{-\frac{\xi^2}{4}} \int_{-\infty}^{\infty} e^{-(x+\frac{\xi}{2})^2} \, dx
\]

The intuition at this point is that sliding the integral from \(-\infty\) to \(+\infty\) along the real axis to be an integral from \(-i\xi - \infty\) to \(-i\xi + \infty\) will not change the value of the integral, since there are no residues to pick up, while it will convert the integrand back to \(e^{-x^2}\), which does not involve \(\xi\).

As usual, an integral from \(-\infty\) to \(+\infty\) is a limit of the corresponding integral from \(-R\) to \(+R\), as \(R \to +\infty\). Then

\[
\int_{-\infty}^{\infty} e^{-(x+\frac{\xi}{2})^2} \, dx = \lim_R \int_{-R}^{R} e^{-(x+\frac{\xi}{2})^2} \, dx = \int_{-\xi - R}^{-i\xi + R} e^{-x^2} \, dx
\]

Let \(B_R\) be the rectangle with vertices \(\pm R\) and \(-i\xi \pm R\), traced counter-clockwise. The integrals over the ends of the box are easily estimated: since \(|e^{-(x+iy)^2}| = e^{-\text{Re}(x+iy)^2} = e^{-x^2+y^2}\),

\[
\left| \int_{-\xi - R}^{-i\xi + R} e^{-x^2} \, dx \right| \leq \text{length} \cdot \text{(sup on curve)} \leq |\xi| \cdot e^{-R^2} \cdot e^{\xi^2} \to 0 \quad \text{as} \ R \to +\infty
\]

Thus,

\[
0 = \lim_{R \to \infty} 0 = \lim_{R \to \infty} \int_{B_R} e^{-ix} e^{-x^2} \, dx = \lim_{R \to \infty} \left( e^{i\xi^2} \int_{-\xi - R}^{-i\xi + R} e^{-x^2} \, dx - e^{-\xi^2} \int_{-R}^{-\infty} e^{-x^2} \, dx \right)
\]
so
\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = e^{-\frac{x^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-x^2} \, dx = e^{-\frac{x^2}{2}} \cdot \sqrt{\pi} \]

This is worth remembering. ///

[03.9] Compute \( \int_{-\infty}^{\infty} e^{-ix} \, dx \)

This is the Fourier transform of \( x \to xe^{-x^2} \). We can reduce it to the previous, slightly simpler, computation by an integration by parts:

\[ \int_{-\infty}^{\infty} e^{-ix} \, dx = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-ix} \frac{d}{dx} e^{-x^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d}{dx} e^{-ix} \cdot e^{-x^2} \, dx \]

Thus,

\[ \int_{-\infty}^{\infty} e^{-ix} \, dx = -\frac{1}{2} i \xi \int_{-\infty}^{\infty} e^{-ix} e^{-x^2} \, dx = -\frac{1}{2} i \xi \cdot e^{-\frac{\xi^2}{4}} \cdot \sqrt{\pi} \]

The integration by parts device is worth remembering. ///

[03.10] For continuous \( \varphi \) on the unit circle \( |z| = 1 \), define

\[ f_\varphi(z) = \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z} \, d\theta \quad \text{ (for } |z| < 1) \]

Show that \( f(z) \) is holomorphic. Give an example of \( \varphi \) not identically 0 so that \( f_\varphi \) is identically 0.

Use Morera’s theorem: with \( \gamma \) be a small counter-clockwise triangle around a given \( z_0 \) in the open unit disk,

\[ \int_\gamma f_\varphi(z) \, dz = \int_0^{2\pi} \varphi(e^{i\theta}) \left( \int_\gamma \frac{dz}{e^{i\theta} - z} \right) d\theta = \int_0^{2\pi} 0 \, d\theta = 0 \]

Morera’s theorem says that this vanishing implies holomorphy of \( f_\varphi \). ///

Note that the given integral is not quite a written-out version of Cauchy’s kernel, because \( d(e^{i\theta}) = i\theta \, e^{i\theta} \, d\theta \), so a factor of \( e^{i\theta} \) is missing. Nevertheless, it’s close. Thus, various heuristics might suggest making \( \varphi(e^{i\theta}) \) be the boundary value of an anti-holomorphic function such as \( F(z) = \bar{z} \). Thus, \( \varphi(e^{i\theta}) = F(e^{i\theta}) = e^{-i\theta} \). For \( |z| < 1 \), expanding a geometric series:

\[ f_\varphi(z) = \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z} \, d\theta = \int_0^{2\pi} \frac{e^{-i\theta}}{e^{i\theta} - z} \, d\theta = \int_0^{2\pi} \frac{e^{-i\theta}}{1 - ze^{-i\theta}} \, d\theta = \sum_{n=0}^{\infty} \int_0^{2\pi} e^{-2i\theta} \left( ze^{-i\theta} \right)^n \, d\theta \]

\[ = \sum_{n=0}^{\infty} z^n \int_0^{2\pi} e^{-i(2n+1)\theta} \, d\theta = \sum_{n=0}^{\infty} z^n \cdot 0 = 0 \]

Thus, with hindsight, \( \varphi(e^{i\theta}) = 1 \) would also have given \( f_\varphi = 0 \). ///

[03.11] Show that a real-valued holomorphic function is constant.

Discussion: There are several possible arguments. First, via Cauchy-Riemann equations: for \( f \) real-valued on a neighborhood of \( z_0 \), taking a derivative along a real direction, but also along a purely imaginary direction, gives

\[ f'(z_0) = \lim_{\varepsilon \to 0} \frac{f(z_0 + \varepsilon) - f(z_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{f(z_0 + i\varepsilon) - f(z_0)}{i\varepsilon} \quad \text{(with } \varepsilon \text{ real)} \]
The first limit is real, the second imaginary, so the equality implies that they are both 0. Thus, \( f' = 0 \), and \( f \) is constant. ///

Second, we can use the open mapping theorem: the real line contains no (non-empty) open sets of \( \mathbb{C} \), so a real-valued holomorphic functions must be constant. ///

Another kind of argument, applicable to \textit{entire} functions with constrained values: for \( f \) entire and real-valued, the function \( F(z) = e^{if(z)} \) takes values on the unit circle. In particular, \( F \) is \textit{bounded} and entire, so \textit{constant}, by Liouville. Then \( 0 = F'(z) = if'(z) e^{if(z)} \), so \( f'(z) = 0 \), and \( f \) is constant. ///

\[03.12\] Show that a holomorphic function \( f \) with \( |f(z)| = 1 \), for all \( z \), is constant.

\textbf{Discussion:} The open mapping succeeds: the unit circle contains no (non-empty) open subsets of \( \mathbb{C} \), so any such \( f \) is constant. ///

\[03.13\] Show that a holomorphic function on \( \mathbb{C} \) taking values in the upper half-plane is constant.

\textbf{Discussion:} Again, the inverse Cayley map \( C^{-1}(z) = \frac{-iz-i}{iz+1} \) (or similar) maps the upper half-plane to the unit disk. It is a holomorphic map, and compositions of holomorphic are holomorphic (because the same is true of complex-differentiable maps), so \( C^{-1} \circ f \) is entire. It is bounded, because it takes values in the unit disk, so by Liouville it is constant. ///

\[03.14\] Let \( C \) be the usual Cantor set

\[ C = \{x \in [0,1] : \text{the ternary expansion of } x \text{ contains only digits 0 and 2, digit 1} \} \]

where terminal repeating 1’s (\ldots11111\ldots) are converted to \ldots2. Show that there is no non-constant holomorphic function with real part taking values in \( C \).

\textbf{Discussion:} One decisive approach is to invoke the open mapping theorem: images of opens under non-constant holomorphic functions are open. The Cantor set contains no non-empty open subsets. For that matter, we have already observed that any \( \mathbb{R} \)-valued holomorphic function is constant, for the same reasons. ///

\[03.15\] Let \( f \) be an entire function such that \( f(z+1) = f(z) \) and \( f(z+i) = f(z) \) for all \( z \). Show that \( f \) is constant.

\textbf{Discussion:} First, the given \textit{periodicity relations} imply that all the values of \( f \) are determined by its values on \( R = \{z = x + iy : 0 \leq x \leq 1, \ 0 \leq y \leq 1\} \): given \( x, y \), there are unique integers \( m, n \) such that \( m \leq x < m+1 \) and \( n \leq y < n+1 \). By the obvious induction,

\[ f(x + iy) = f((x - m) + i(y - n)) \]

while \( 0 \leq x - m < 1 \) and \( 0 \leq y - n < 1 \). On the compact set \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \), the continuous function \( f \) is \textit{bounded}. Thus, \( f \) is entire and bounded, so by Liouville, it is constant. ///