Weierstrass and Hadamard products

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1. Weierstrass products
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Apart from factorization of polynomials, after Euler’s

\[
\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)
\]

there is Euler’s product for \(\Gamma(z)\), which he used as the definition of the Gamma function:

\[
\int_{0}^{\infty} e^{-t} t^z \frac{dt}{t} = \Gamma(z) = \frac{1}{z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}}
\]

where the Euler-Mascheroni constant \(\gamma\) is essentially defined by this relation. The integral (Euler’s) converges for \(\text{Re}(z) > 0\), while the product (Weierstrass’) converges for all complex \(z\) except non-positive integers.

Because the exponential factors are linear, and can cancel,

\[
\frac{1}{\Gamma(z) \cdot \Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{z}{\pi} \cdot \sin \pi z
\]

Linear exponential factors are exploited in Riemann’s explicit formula [Riemann 1859], derived from equality of the Euler product and Hadamard product [Hadamard 1893] for the zeta function \(\zeta(s) = \sum_n \frac{1}{n^s}\) for \(\text{Re}(s) > 1\):

\[
\prod_{\rho \text{ prime}} \frac{1}{1 - \rho^{-s}} = \zeta(s) = \frac{e^{a+bs}}{s-1} \cdot \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{ho s} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right) e^{-s/2n}
\]

where the product expansion of \(\Gamma(\frac{z}{2})\) is visible, corresponding to trivial zeros of \(\zeta(s)\) at negative even integers, and \(\rho\) ranges over all other, non-trivial zeros, known to be in the critical strip \(0 < \text{Re}(s) < 1\).

The hard part of the proof (below) of Hadamard’s theorem is essentially that of [Ahlfors 1953/1966], with various rearrangements. A somewhat different argument is in [Lang 1993]. Some standard folkloric proofs of supporting facts about harmonic functions are recalled.
1. Weierstrass products

Given a sequence of complex numbers $z_j$ with no accumulation point in $\mathbb{C}$, we will construct an entire function with zeros exactly the $z_j$.

[1.1] Basic construction

Taylor-MacLaurin polynomials of $-\log(1 - z)$ will play a role: let

$$p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^n}{n}$$

We will exhibit a sequence of integers $n_j$ giving an absolutely convergent infinite product vanishing precisely at the $z_j$, with vanishing at $z = 0$ accommodated by a suitable leading factor $z^m$, of the form

$$z^m \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_{n_j}(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^{n_j}}{n_jz_j^{n_j}}\right)$$

Absolute convergence of $\sum_j \log(1 + a_j)$ implies absolute convergence of the infinite product $\prod_j (1 + a_j)$. Thus, we show that

$$\sum_j \left| \log \left(1 - \frac{z}{z_j}\right) + p_{n_j} \left(\frac{z}{z_j}\right) \right| < \infty$$

Fix a large radius $R$, keep $|z| < R$, and ignore the finitely-many $z_j$ with $|z_j| < 2R$, so in the following $|z/z_j| < \frac{1}{2}$. Using the power series expansion of $\log$,

$$\left| \log(1 - \frac{z}{z_j}) - p_n \left(\frac{z}{z_j}\right) \right| \leq \frac{1}{n+1} \cdot \left|\frac{z}{z_j}\right|^{n+1} + \frac{1}{n+2} \cdot \left|\frac{z}{z_j}\right|^{n+2} + \ldots \leq \frac{1}{n+1} \cdot \frac{|z/z_j|^{n+1}}{1 - |z/z_j|} \leq 2 \cdot \frac{|z/z_j|^{n+1}}{n+1}$$

Thus, we want a sequence of positive integers $n_j$ such that

$$\sum_{|z_j| \geq 2R} \frac{|z/z_j|^{n+1}}{n_j + 1} < \infty \quad \text{(with } |z| < R)$$

The choice of $n_j$’s must be compatible with enlarging $R$, and this is easily arranged. For example, $n_j = j - 1$ succeeds:

$$\sum_j |\frac{z}{z_j}|^j = \sum_{|z_j| < 2R} |\frac{z}{z_j}|^j + \sum_{|z_j| \geq 2R} |\frac{z}{z_j}|^j \leq \sum_{|z_j| < 2R} |\frac{z}{z_j}|^j + \sum_j 2^{-j}$$

Since $\{z_j\}$ is discrete, the sum over $|z_j| < 2R$ is finite, giving convergence, and convergence of the infinite product with $n_j = j$:

$$\prod_j \left(1 - \frac{z}{z_j}\right) e^{p_j(z/z_j)} = \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^j}{jz^j}\right)$$

[1.2] Canonical products and genus

Given entire $f$ with zeros $z_j \neq 0$ and a zero of order $m$ at 0, ratios

$$\varphi(z) = \frac{f(z)}{z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_{n_j}(z/z_j)}}$$
with convergent infinite products are entire, and do not vanish. Non-vanishing entire $\varphi$ has an entire logarithm:
\[
g(z) = \log \varphi(z) = \int_0^z -\frac{\varphi'(\zeta)}{\varphi(\zeta)} \, d\zeta
\]
Thus, non-vanishing entire $\varphi$ is expressible as
\[
\varphi(z) = e^{g(z)} \quad \text{(with $g$ entire)}
\]
Thus, the most general entire function with prescribed zeros is of the form
\[
f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_n_j(z/z_j)} \quad \text{(with $g$ entire)}
\]
With fixed $f$, altering the $n_j$ necessitates a corresponding alteration in $g$.

We are most interested in zeros $\{z_j\}$ allowing a uniform integer $h$ giving convergence of the infinite product in an expression
\[
f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_n_j(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp \left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^h}{hz_j^h}\right)
\]
When $f$ admits such a product expression with a uniform $h$, a product expression with minimal uniform $h$ is a canonical product for $f$.

When, further, the leading factor $e^{g(z)}$ for $f$ has $g(z)$ a polynomial, the genus of $f$ is the maximum of $h$ and the degree of $g$.

2. Poisson-Jensen formula

Jensen’s formula and the Poisson-Jensen formula are essential in the difficult half of the Hadamard theorem (below) comparing genus of an entire function to its order of growth.

The logarithm $u(z) = \log |f(z)|$ of the absolute value $|f(z)|$ of a non-vanishing holomorphic function $f$ on a neighborhood of the unit disk is harmonic, that is, is annihilated by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$: expand
\[
\Delta \log |f(z)| = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \left(\frac{1}{2} \log f(z) + \frac{1}{2} \log F(z)\right)
\]
Conveniently, the two-dimensional Laplacian is the product of the Cauchy-Riemann operator and its conjugate. Since $\log f$ is holomorphic and $\log F$ is anti-holomorphic, both are annihilated by the product of the two linear operators, so $\log |f(z)|$ is harmonic.

Thus, $\log |f(z)|$ satisfies the mean-value property for harmonic functions
\[
\log |f(0)| = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta
\]
Next, let $f$ have zeros $\rho_j$ in $|z| < 1$ but none on the unit circle. We manufacture a holomorphic function $F$ from $f$ but without zeros in $|z| < 1$, and with $|F| = |f|$ on $|z| = 1$, by the standard ruse
\[
F(z) = f(z) \cdot \prod_j \frac{1 - \overline{\rho_j} z}{z - \rho_j}
\]
Indeed, for $|z| = 1$, the numerator of each factor has the same absolute value as the denominator:

$$|z - \rho_j| = \left| \frac{1}{z} - \frac{1}{\bar{\rho}_j} \right| = \frac{1}{|z|} \cdot |1 - \bar{\rho}_j z| = \left| 1 - \bar{\rho}_j z \right|$$

For simplicity, suppose no $\rho_j$ is 0. Applying the mean-value identity to $\log |F(z)|$ gives

$$\log |f(0)| - \sum_j \log |\rho_j| = \log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta$$

and then the basic Jensen’s formula

$$\log |f(0)| - \sum_j \log |\rho_j| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta \quad \text{(for } |\rho_j| < 1)$$

The Poisson-Jensen formula is obtained by replacing 0 by an arbitrary point $z$ inside the unit disk, by replacing $f$ by $f \circ \varphi_z$, where $\varphi_z$ is a linear fractional transformation mapping 0 to $z$ and stabilizing[1] the unit disk:

$$\varphi_z = \left( \frac{1}{\bar{z}} \quad z \right) : w \rightarrow \frac{w + z}{\bar{z}w + 1}$$

This replaces the zeros $\rho_j$ by $\varphi^{-1}_z(\rho_j) = \frac{\rho_j - z}{\rho_j z + 1}$. Instead of the mean-value property expressing $f(0)$ as an integral over the circle, use the Poisson formula for $f(z)$. This gives the basic Poisson-Jensen formula

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\bar{\rho}_j z + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < 1, |\rho_j| < 1)$$

Generally, for holomorphic $f$ on a neighborhood of a disk of radius $r > 0$ with zeros $\rho_j$ in that disk, apply the previous to $f(r \cdot z)$ with zeros $\rho_j/r$ in the unit disk:

$$\log |f(r \cdot z)| - \sum_j \log \left| \frac{\rho_j / r - z}{-\bar{\rho}_j / r + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r \cdot e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < 1)$$

Replacing $z$ by $z/r$ gives

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j / r - z / r}{-\bar{\rho}_j / r + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{1 - |z/r|^2}{|z/r - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < r)$$

which rearranges to the general Poisson-Jensen formula

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\bar{\rho}_j z + r + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < r, |\rho_j| < r)$$

[1] To verify that such maps stabilize the unit disk, expand the natural expression:

$$1 - \left| \frac{w + z}{\bar{z}w + 1} \right|^2 = |\bar{z}w + 1|^{-2} \cdot \left( |\bar{z}w + 1|^2 - |w + z|^2 \right) = |\bar{z}w + 1|^{-2} \cdot \left( |zw|^2 + \bar{z}w + z\bar{w} + 1 - |w|^2 - \bar{w} - z| \right)$$

$$= |\bar{z}w + 1|^{-2} \cdot \left( |zw|^2 + 1 - |w|^2 - |z|^2 \right) = |\bar{z}w + 1|^{-2} \cdot (1 - |z|^2) \cdot (1 - |w|^2) > 0$$
The case \( z = 0 \) is the general Jensen formula for arbitrary radius \( r \): 
\[
\log |f(0)| - \sum_j \log \left| \frac{\rho_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \quad \text{(with } |\rho_j| < r)\]

3. Hadamard products

The order of an entire function \( f \) is the smallest positive real \( \lambda \), if it exists, such that, for every \( \varepsilon > 0 \), 
\[
|f(z)| \leq e^{|z|^{\lambda+\varepsilon}} \quad \text{(for all sufficiently large } |z|)\]

Recall that, in an infinite product expression with compensating exponential factors with uniform degree \( h \)
\[
f(z) = e^{g(z)} \cdot z^m \prod_j \left( 1 - \frac{z}{z_j} \right) e^{p_h(z/z_j)} = z^m \prod_j \left( 1 - \frac{z}{z_j} \right) \exp \left( \frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^h}{hz^h} \right)\]

when the exponent \( g(z) \) is polynomial, the genus of \( f \) is the maximum of \( h \) and the degree of \( g \).

\[3.0.1\] Theorem: (Hadamard) The genus \( h \) and order \( \lambda \) are related by \( h \leq \lambda < h + 1 \). In particular, one is finite if and only the other is finite.

Proof: First, the easier half. For \( f \) of finite genus \( h \) expressed as
\[
f(z) = e^{g(z)} \cdot z^m \prod_j \left( 1 - \frac{z}{z_j} \right) e^{p_h(z/z_j)} = e^{g(z)} \cdot z^m \prod_j \left( 1 - \frac{z}{z_j} \right) \exp \left( \frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^h}{hz^h} \right)\]

the leading exponent is polynomial \( g \) of degree at most \( h \), so \( e^{g(z)} \) is of order at most \( h \). The order of a product is at most the maximum of the orders of the factors, so it suffices to prove that the order of the infinite product is at most \( h + 1 \).

The assumption that \( h \) is the genus of \( f \) is equivalent to 
\[
\sum_j \frac{1}{|z_j|^{h+1}} < \infty
\]

We use this to directly estimate the infinite product and show that it has order of growth \( \lambda < h + 1 \).

Toward an estimate on \( F_h(w) = (1 - w) e^{p_h(w)} \) applicable for all \( w \), not merely for \( |w| < 1 \), we collect some inequalities. There is the basic
\[
\log |F_h(w)| = \log |(1 - w) e^{p_{h-1}(w)} \cdot e^{w^h/h}| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \quad \text{(for all } w)\]

As before, for \( |w| < 1 \),
\[
\log |F_h(w)| \leq \frac{1}{h+1} \cdot |w|^{h+1} + \frac{1}{h+2} \cdot |w|^{h+2} + \ldots \leq |w|^{h+1} \cdot \frac{1}{1 - |w|} \quad \text{(for } |w| < 1)\]

This gives \((1 - |w|) \cdot \log |F_h(w)| \leq |w|^{h+1} \) for \( |w| < 1 \). Adding to the latter the basic relation multiplied by \( |w| \) gives
\[
\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + (1 + \frac{1}{h}) |w|^{h+1} \quad \text{(for } |w| < 1)\]
In fact, the latter inequality also holds for $|w| \geq 1$ and $\log |F_{h-1}(w)| \geq 0$, from the basic relation. For $\log |F_{h-1}(w)| < 0$ and $|w| \geq 1$, from the basic relation,

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \leq \frac{|w|^h}{h} \leq (1 + \frac{1}{h})|w|^{h+1} \quad (\text{for } \log |F_{h-1}(w)| < 0 \text{ and } |w| \geq 1)$$

Now prove $\log |F_h(w)| \ll_h |w|^{h+1}$, by induction on $h$. For $h = 0$, from $\log |x| \leq |x| - 1$,

$$\log |1 - w| \leq |1 - w| - 1 \leq 1 + |w| - 1 = |w|$$

Assume $\log |F_{h-1}(w)| \ll_h |w|^h$. For $|w| < 1$, we reach the desired conclusion by

$$\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + (1 + \frac{1}{h})|w|^{h+1} \ll_h |w| \cdot |w|^h + (1 + \frac{1}{h})|w|^{h+1} \quad (\text{for } |w| < 1)$$

For $|w| \geq 1$ and $\log |F_{h-1}(w)| > 0$, from the basic relation

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \ll_h |w|^h + \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad (\text{for } |w| \geq 1 \text{ and } \log |F_{h-1}(w)| > 0)$$

For $\log |F_{h-1}(w)| \leq 0$ and $|w| \geq 1$, from the basic relation we already have

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \ll_h |w|^h \ll_h |w|^{h+1} \quad (\text{for } |w| \geq 1 \text{ and } \log |F_{h-1}(w)| < 0)$$

This proves $\log |F_h(w)| \ll_h |w|^{h+1}$ for all $w$.

Estimate the infinite product:

$$\log \prod_j \left(1 - \frac{z}{z_j} \right) e^{p_h(z_j)} = \sum_j \log \left(1 - \frac{z}{z_j} \right) e^{p_h(z_j)} \ll_h \sum_j \left| \frac{z}{z_j} \right|^{h+1} < \infty$$

since $\sum \left| \frac{z}{z_j} \right|^{h+1}$ converges. Thus, such an infinite product has growth order $\lambda \leq h + 1$.

Now the difficult half of the proof. Let $h \leq \lambda < h + 1$. Jensen’s formula will show that the zeros $z_j$ are sufficiently spread out for convergence of $\sum 1/|z_j|^{h+1}$. Without loss of generality, suppose $f(0) \neq 0$. From

$$\log |f(0)| - \sum_j \log \left| \frac{z_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \quad (\text{with } |z_j| < r)$$

certainly

$$\sum_{|z_j| < r/2} \log |z_j| \leq \sum_{|z_j| < r/2} - \log \left| \frac{r_j}{r} \right| \ll \varepsilon \cdot \log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} r^{\lambda + \varepsilon} \, d\theta \ll r^{\lambda + \varepsilon} \quad (\text{for every } \varepsilon > 0)$$

With $\nu(r)$ the number of zeros inside the disk of radius $r$, this gives

$$\lim_{r \to +\infty} \frac{\nu(r)}{r^{\lambda + \varepsilon}} = 0 \quad (\text{for all } \varepsilon > 0)$$

Order the zeros by absolute value: $|z_1| \leq |z_2| \leq \ldots$ and for simplicity suppose no two have the same size. Then $j = \nu(|z_j|) \ll \varepsilon |z|^\lambda + \varepsilon$. Thus,

$$\sum_{|z_j|^{h+1}} \ll \sum_j \left( \frac{1}{j^{\lambda+\varepsilon}} \right)^{h+1} = \sum_j \frac{1}{j^{\lambda+\varepsilon}}$$
The latter converges for $\frac{h+1}{\lambda+\varepsilon} > 1$, that is, for $\lambda + \varepsilon < h + 1$. When $\lambda < h + 1$, there is $\varepsilon > 0$ making such an equality hold.

It remains to show that the entire function $g(z)$ in the leading exponential factor is of degree at most $h + 1$, by showing that its $(h+1)^{th}$ derivative is 0.

In the Poisson-Jensen formula

$$\log |f(z)| - \sum_{|z_j| < r} \log \left| \frac{z_j - z}{\pi z_j / r + r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} d\theta$$

(for $|z| < r$)

application of $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ annihilates the anti-holomorphic parts, returning us to an equality of holomorphic functions, as follows. The effect on the integrand is

$$2 \left( \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} = 2 \frac{-\pi}{(z - re^{i\theta})(\pi - re^{-i\theta})} - \frac{r^2 - |z|^2}{(z - re^{i\theta})^2(\pi - re^{-i\theta})}$$

$$= 2 \frac{-|z|^2 + \pi re^{i\theta} - r^2 + |z|^2}{(z - re^{i\theta})^2(\pi - re^{-i\theta})} = 2 \frac{re^{i\theta}}{(z - re^{i\theta})^2}$$

Thus,

$$\frac{f'(z)}{f(z)} = \sum_{|z_j| < r} \frac{1}{z - z_j} + \sum_{|z_j| < r} \frac{\pi_j}{z_j z - r^2} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^2} d\theta$$

Further differentiation $h$ times in $z$ gives

$$\left(\frac{f'(z)}{f(z)}\right)^{(h)} = \sum_{|z_j| < r} \frac{(-1)^h h!}{(z - z_j)^{h+1}} - \sum_{|z_j| < r} \frac{(-1)^h h! \pi_j^{h+1}}{(z_j z - r^2)^{h+1}} + \frac{(-1)^h (h+1)!}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta$$

We claim that the second sum and the integral go to 0 as $r \to +\infty$.

Regarding the integral, Cauchy’s integral formula gives

$$\int_0^{2\pi} \frac{re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta = 0$$

Letting $M_r$ be the maximum of $|f|$ on the circle of radius $r$, taking $|z| < r/2$, up to sign the integral is

$$\int_0^{2\pi} \log \left( \frac{M_r}{|f(re^{i\theta})|} \right) \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta \ll \frac{1}{r^{h+1}} \int_0^{2\pi} \log \left( \frac{M_r}{|f(re^{i\theta})|} \right) d\theta \ll \frac{1}{r^{\lambda+\varepsilon}} \int_0^{2\pi} - \log |f(re^{i\theta})| d\theta$$

Jensen’s formula gives

$$\frac{1}{2\pi} \int_0^{2\pi} - \log |f(re^{i\theta})| d\theta \leq - \log |f(0)|$$

Thus, for $\lambda + \varepsilon < h + 1$ the integral goes to 0 as $r \to +\infty$.

For the second sum, again take $|z| < r/2$, so for $|z_j| < r$

$$\left| \frac{\pi_j^{h+1}}{(z_j z - r^2)^{h+1}} \right| \leq \frac{|z_j^{h+1}|}{r^{h+1}(r - |z_j|)^{h+1}} \leq \frac{|z_j^{h+1}|}{r^{h+1}}$$

We already showed that the number $\nu(r)$ of $|z_j| < r$ satisfies $\lim \nu(r)/r^{h+1} = 0$. Thus, this sum goes to 0 as $r \to +\infty$. Taking the limit,

$$\left( \frac{f'}{f} \right)^{(h)} = (-1)^h h! \sum_j \frac{1}{(z - z_j)^{h+1}}$$
Returning to \( f(z) = e^{g(z)} \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_n(z/z_j)} \), taking logarithmic derivative gives

\[
\frac{f'}{f} = g' + \sum_j \left( \frac{1}{z - z_j} + \frac{p'_n(z/z_j)}{z_j} \right)
\]

and taking \( h \) further derivatives gives

\[
\left( \frac{f'}{f} \right)^{(h)} = g^{(h+1)} + \sum_j \frac{(-1)^h h!}{(z - z_j)^{h+1}}
\]

Since the \( h^{th} \) derivative of \( f'/f \) is the latter sum, \( g^{(h+1)} = 0 \), so \( g \) is a polynomial of degree at most \( h \).


