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Hartogs' Theorem: separate analyticity implies joint

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(The present proof of this old result roughly follows the proof given in Hörmander's *An Introduction to Complex Analysis in Several Variables*, which I believe roughly follows Hartogs' original argument.)

Theorem: Let f be a $\mathbf{C}$-valued function defined in an open set $U \subset \mathbf{C}^n$. Suppose that f is analytic in each variable z_j when the other coordinates z_k for $k \neq j$ are fixed. Then f is analytic as a function of all n coordinates.

Remark: Absolutely no additional hypothesis on f is used beyond its separate analyticity. Specifically, there is no assumption of continuity, nor even of measurability. Indeed, the beginning of the proof illustrates the fact that an assumption of continuity trivializes things. The strength of the theorem is that no hypothesis whatsoever is necessary.

Proof: The assertion is local, so it suffices to prove it when the open set U is a polydisk. The argument approaches the full assertion in stages.

First, suppose that f is *continuous* on the closure $\bar{U}$ of a polydisk U, and separately analytic. Even without continuity, simply by separate analyticity, an n-fold iterated version of Cauchy's one-variable integral formula is valid, namely

$$f(z) = \frac{1}{(2\pi i)^n} \int_{C_1} \dots \int_{C_n} \frac{f(\zeta)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)}\, d\zeta_1 \dots d\zeta_n$$

where C_j is the circle bounding the disk in which z_j lies, traversed in the positive direction. The integral is a compactly supported integral of the function

$$(\zeta_1, \dots, \zeta_n) \to \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1)\dots(\zeta_n - z_n)}$$

For $|z_j| < |\zeta_j|$, the geometric series expansion

$$\frac{1}{\zeta_j - z_j} = \sum_{n \geq 0} \frac{z_j^n}{\zeta_j^{n+1}}$$

can be substituted into the latter integral. Fubini's theorem justifies interchange of summation and integration, yielding a (convergent) power series for $f(z)$. Thus, continuity of $f(z)$ (with separate analyticity) implies joint continuity.

Note that if we could be sure that *every* conceivable integral of analytic functions were analytic, then this iterated one-variable Cauchy formula would prove (joint) analyticity immediately. However, it is not obvious that *separate* analyticity implies continuity, for example.

Next we see that *boundedness* of a separately analytic function on a closed polydisk implies continuity, using Schwarz' lemma and its usual corollary:

Lemma: (Schwarz) Let $g(z)$ be a holomorphic function on $\{z \in \mathbf{C} : |z| < 1\}$, with $g(0) = 0$ and $|g(z)| \leq 1$. Then $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$. (*Proof:* Apply the maximum modulus principle to $f(z)/z$ on disks of radius less than 1.)

Corollary: Let $g(z)$ be a holomorphic function on $\{z \in \mathbf{C} : |z| < r\}$, with $|g(z)| \leq B$ for a bound B. Then for z, ζ in that disk,

$$|g(z) - g(\zeta)| \leq 2 \cdot B \cdot \left| \frac{r(z - \zeta)}{r^2 - \bar{\zeta} z} \right|$$

Proof: (of corollary) The linear fractional transformation

$$\mu : z \to r \cdot \begin{pmatrix} 1 & \zeta/r \\ \bar{\zeta}/r & 1 \end{pmatrix} (rz) = r \cdot \frac{z + r\zeta}{\bar{\zeta} z + r}$$

sends the disk of radius 1 to the disk of radius r, and sends 0 to ζ. Then the function

$$z \to \frac{g(\mu(z)) - g(\zeta)}{2B}$$

is normalized to match Schwarz' lemma, namely that it vanishes at 0, and is bounded by 1 on the open unit disk. Thus, we conclude that for $|z| < 1$

$$\left| \frac{g(\mu(z)) - g(\zeta)}{2B} \right| \leq |z|$$

Replace z by

$$\mu^{-1}(z) = \frac{r(z-\zeta)}{r^2 - \bar{\zeta} z}$$

to obtain

$$\left| \frac{g(z) - g(\zeta)}{2B} \right| \leq \left| \frac{r(z-\zeta)}{r^2 - \bar{\zeta} z} \right|$$

as asserted in the corollary. ///

Now let f be separately analytic and *bounded* on the closure of the polydisk $\{(z_1, \ldots, z_n) : |z_j| < r_j\}$. We show that f is (jointly) analytic by proving it is continuous, invoking the first part of the proof (above). Let B be a bound for $|f|$ on the closed polydisk. We claim that the inequality

$$|f(z) - f(\zeta)| \leq 2B \sum_{1 \leq j \leq n} \frac{r_j |z_j - \zeta_j|}{|r_j^2 - \bar{\zeta}_j z_j|}$$

holds, which would prove continuity. Because of the telescoping expression

$$f(z) - f(\zeta) = \sum_{1 \leq j \leq n} \left(f(\zeta_1, \ldots, \zeta_{j-1}, z_j, \ldots, z_n) - f(\zeta_1, \ldots, \zeta_j, \zeta_j, z_{j+1}, \ldots, z_n) \right)$$

it suffices to prove the inequality in the single-variable case, which is the immediate corollary to Schwarz' lemma as above. Thus, a *bounded* separately analytic f is continuous, and (from above) jointly analytic.

Now we do induction on the dimension n: suppose that Hartogs' theorem is proven on $\mathbf{C}^{n-1}$, and prove it for $\mathbf{C}^n$. Here the Baire Category Theorem intervenes, getting started on the full statement of the theorem by first showing that a separately analytic function must be bounded on *some* polydisk, hence (from above) *continuous* on that polydisk, hence (from above) *analytic* on that polydisk.

Let f be separately analytic on a (non-empty) closed polydisk $D = \prod_{1 \leq j \leq n} D_j$, where D_j is a disk in $\mathbf{C}$. We claim that there exist non-empty closed disks $E_j \subset D_j$ with $E_n = D_n$ such that f is bounded on $E = \prod_{1 \leq j \leq n} E_j$ (and, hence, f is analytic in E).

To see this, for each bound $B > 0$ let

$$\Omega_B = \{z' \in \prod_{1 \leq j \leq n-1} E_j : |f(z', z_n)| \leq B \text{ for all } z_n \in E_n\}$$

By induction, for fixed z_n the function $z' \to f(z', z_n)$ is analytic, so continuous, so Ω_B is closed. For any fixed z', the function $z_n \to f(z', z_n)$ is assumed analytic, so is continuous on the closed disk $E_n = D_n$, hence bounded. Thus

$$\bigcup_{B=1}^{\infty} \Omega_B = \prod_{1 \leq j \leq n-1} D_j$$

Then the Baire Category Theorem shows that some Ω_B must have non-empty interior, so must contain a (non-empty) closed polydisk, as claimed

Now let f be separately analytic in a polydisk

$$D = \{(z_1, \ldots, z_n) : |z_j| < r\} \subset \mathbf{C}^n$$

analytic in $z' = (z_1, \ldots, z_n - 1)$ for fixed z_n, and suppose that f is analytic in a smaller (non-empty) polydisk

$$E = \left(\prod_{1 \le j \le n-1} \{z_j \in \mathbf{C} : |z_j| < \varepsilon\} \right) \times \{z_n \in \mathbf{C} : |z_n| < r\}$$

inside D. Then we claim that f is analytic on the original polydisk D.

By the iterated form of Cauchy's formula, the function $z' \to f(z', z_n)$ has a Taylor expansion in z'

$$f(z', z) = \sum_\alpha c_\alpha(z_n)\, z'^\alpha$$

where the coefficients depend upon z_n, given by the usual formula

$$c_\alpha(z_n) = \frac{\partial^\alpha}{\partial z'^\alpha} f(0, z_n)/\alpha!$$

using multi-index notation. Cauchy's integral formula in z' for derivatives

$$\frac{\partial^\alpha}{\partial z'^\alpha} f(0, z_n) = \alpha!\, \frac{1}{(2\pi i)^{n-1}} \int_{C_1} \cdots \int_{C_{n-1}} \frac{f(\zeta)}{(\zeta_1 - z_1)^{\alpha_1+1} \ldots (\zeta_{n-1} - z_{n-1})^{\alpha_{n-1}+1}}\, d\zeta_1 \ldots d\zeta_{n-1}$$

shows that $c_\alpha(z_n)$ is analytic in z_n, again by expanding convergent geometric series and their derivatives, and interchanging summation and integration.

Fix $0 < r_1 < r_2 < r$ and fix z_n with $|z_n| < r$. Then

$$|c_\alpha(z_n)| \cdot r_2^{|\alpha|} \to 0$$

as $|\alpha| \to \infty$, by the convergence of the power series. Let B be a bound for $|f|$ on the smaller polydisk E. Then on that smaller polydisk the Cauchy integral formula for the derivative gives

$$|c_\alpha(z_n)| \le B/\varepsilon^{|\alpha|}$$

Therefore, the subharmonic functions

$$u_\alpha(z_n) = \frac{1}{|\alpha|} \log |c_\alpha(z_n)|$$

are uniformly bounded from above for $|z_n| < r$. And the property $|c_\alpha(z_n)| \cdot r_2^{|\alpha|} \to 0$ shows that for fixed z_n $\log(1/r_2)$ is an upper bound for these subharmonic functions as $|\alpha| \to \infty$. Thus, Hartogs' lemma (recalled below) on subharmonic functions implies that for large $|\alpha|$, *uniformly* in $|z_n| < r_1$

$$\frac{1}{|\alpha|} \log |c_\alpha(z_n)| \le \log(1/r_1)$$

Thus, for large $|\alpha|$

$$|c_\alpha(z_n)| \cdot r_1^{|\alpha|} \le 1$$

uniformly in $|z_n| < r_1$. Therefore, since the summands $c_\alpha(z_n)\, z'^\alpha$ are analytic, the series

$$f(z', z) = \sum_\alpha c_\alpha(z_n)\, z'^\alpha$$

converges to a function analytic in the polydisk D.

Thus, in summary, given $z \in U$, choose $r > 0$ so that the polydisk of radius $2r$ centered at z is contained in U. The Baire category argument above shows that there is w such that z is inside a polydisk D of radius r centered at w, and such that f is holomorphic on some smaller polydisk E inside D (still centered at w). Finally one uses Hartogs' lemma on subharmonic functions (below) to see that the power series for f on the small polydisk E at w converges on the larger polydisk D at w. Since D contains the given point z, f is analytic on a neighborhood of z. Thus, f is analytic throughout U. ///

Lemma: (Hartogs) Let u_j be a sequence of real-valued subharmonic functions in an open set U in $\mathbf{C}$. Suppose that the functions are uniformly bounded from above, and that

$$\limsup_k u_k(z) \leq C$$

for every $z \in U$. Then, given $\varepsilon > 0$ and compact $K \subset U$ there exists k_o such that for $z \in K$ and $k \geq k_o$

$$u_k(z) \leq C + \varepsilon$$

Proof: Without loss of generality, replacing U by an open subset with compact closure contained inside U, we may suppose that the functions u_k are uniformly bounded in U, for example $u_k(z) \leq 0$ for all $z \in U$. Let $r > 0$ be small enough so that the distance from K to every point of the complement of U is more than $3r$. Using the proposition below characterizing subharmonic functions, we have, for every $z \in K$,

$$\pi r^2 u_k(z) \leq \int_{|z-\zeta|<r} u_k(\zeta)\, d\zeta$$

By Fatou's lemma, the lim sup of the right hand side is at most $\pi r^2 C$ as $k \to \infty$. Thus, for every $z \in K$ there is k_o such that for $k \geq k_o$

$$\int_{|z-\zeta|<r} u_k(\zeta)\, d\zeta \leq \pi r^2 (C + \varepsilon/2)$$

Since $u_k(z) \leq 0$, for $|z - w| < \delta < r$

$$\pi(r+\delta)^2\, u_k(w) \leq \int_{|\zeta-w|<r+\delta|} u_k(\zeta)\, d\zeta \leq \int_{|\zeta-z|\leq r} u_k(\zeta)\, d\zeta$$

Thus, for $\delta > 0$ sufficiently small, for $k \geq k_o$ and $|w - z| < \delta$,

$$u_k(w) < C + \varepsilon$$

Since K is compact the lemma follows. ///

For convenience, we recall the following basic property of subharmonic functions.

Proposition: For a real-valued subharmonic function u bounded above on an open set U, for every positive measure μ on $[0, \delta]$, and for $z \in U$ of distance more than δ from the complement of U,

$$u(z) \cdot 2\pi \cdot \int d\mu \leq \int_0^{2\pi} \int u(z + re^{i\theta})\, d\theta\, d\mu(r)$$

Proof: The definition of a function u being subharmonic on an open set Ω is that u is upper semicontinuous (that is, $\{z \in \Omega : u(z) < c\}$ is open for every constant c), and for every compact $K \subset \Omega$, for every continuous function h on K harmonic on K and $h(\beta) \geq u(\beta)$ for β on the boundary of K, $u(z) \leq h(z)$ throughout K. The condition may be vacuous unless u is assumed bounded from above.

Let $z \in U$ be distance more than δ away from the complement of U, and fix r with $0 < r \le \delta$. Let D be the closed disk of radius r about z. Since $r \le \delta$, $D \subset U$. For a trigonometric polynomial

$$g(\theta) = \sum_k c_k\, e^{i\theta}$$

with real coefficients c_k with $u(z + re^{i\theta}) \le g(\theta)$, the polynomial

$$G(\zeta) = c_0 + \sum_{k>0} (c_k + c_{-k})\, \frac{(\zeta - z)^k}{r^k}$$

has real part $\mathrm{Re}G$ which is an upper bound for u on the boundary of the disk D. Thus, $u \le \mathrm{Re}G$ on D by the subharmonicness of u, and in particular at the center of D, at z,

$$u(z) \le c_o + \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\, d\theta$$

Then for an arbitrary continuous real-valued function h on the boundary of D and with $u(z + re^{i\theta}) \le h(\theta)$, (by Weierstrass approximation, for example) given $\varepsilon > 0$ we can find a trigonometric polynomial g so that $\sup |g(\theta) - h(\theta)| < \varepsilon$. Thus, for every $\varepsilon > 0$,

$$u(z) \le c_o + \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\, d\theta + \varepsilon$$

Thus, the latter inequality must hold with $\varepsilon = 0$, for continuous h. Since the integral of an upper-semicontinuous function is the infimum of the integrals of continuous functions dominating it, we have the same inequality with u in place of h. Integration with respect to the radius r gives the result.
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In fact, suppose that for every $\delta > 0$ and for every z at distance more than δ from the complement of U there exists a positive measure μ on $[0, \delta]$ with support not just $\{0\}$ and

$$u(z) \cdot 2\pi \cdot \int d\mu \le \int_0^{2\pi} \int u(z + re^{i\theta})\, d\theta\, d\mu(r)$$

Then u is subharmonic. To see this, let K be a compact subset of U, h a continuous function on K which is harmonic in the interior of K and such that $u \le g$ on the boundary of K. If the supremum of $u - h$ over K is strictly positive, the upper semicontinuity of $u - h$ implies that $u - h$ attains its sup S on a non-empty compact subset M of the interior of K. Let z_o be a point of M closest to the boundary of K. If the distance is greater than δ, then *every* circle $|z - z_o| = r$ with $0 < r \le \delta$ contains a non-empty arc of points where $u - h < S$. Then

$$\int (u-h)(z_o + re^{i\theta}))\, d\theta\, d\mu(r) < S \cdot 2\pi \cdot \int d\mu(r) = (u-h)(z_o) \cdot 2\pi \cdot \int d\mu(r)$$

when μ is a measure not supported just at $\{0\}$. The mean value property for harmonic functions gives

$$\int h(z_o + re^{i\theta}))\, d\theta\, d\mu(r) = h(z_o) \cdot 2\pi \cdot \int d\mu(r)$$

Thus,

$$\int u(z_o + re^{i\theta}))\, d\theta\, d\mu(r) < u(z_o) \cdot 2\pi \cdot \int d\mu(r)$$

contradicting the hypothesis. Thus, $\sup_K (u - h) \le 0$, which proves that u is subharmonic. ///