The exponential function, sine, cosine

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We attempt to solve the differential equation (with initial condition)

\[ u' = u \quad u(0) = 1 \]

in terms of a power series

\[ u(x) = \sum_{n=0}^{\infty} c_n x^n \]

From Abel’s theorem, assuming convergence,

\[ u'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1} \quad u(0) = c_0 \]

and the coefficients are uniquely determined by the values of the function. This implies a sequence of equalities

\[ c_0 = 1, \ 1 \cdot c_1 = c_0, \ 2 \cdot c_2 = c_1, \ 3 \cdot c_3 = c_2, \ldots \]

from which [1] \( c_n = 1/n! \) and

\[ u(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]

This outcome was under an assumption that the series so-obtained is convergent, and, by the ratio test, it is indeed (absolutely) convergent for all real \( x \), and, similarly, for all complex \( x \). Write

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

Note that it converges uniformly absolutely for \( z \) in compact subsets of the complex plane.

[0.0.1] Proposition: For complex \( z, w \),

\[ e^{z+w} = e^z \cdot e^w \]

The complex conjugate \( \overline{e^z} \) of \( e^z \) is \( e^{\overline{z}} \).

Proof: The first assertion is a corollary of the binomial theorem, as follows. With absolute convergence justifying rearrangements:

\[ e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n,i} \binom{n}{i} \frac{z^{n-i}w^i}{n!} \]

with usual binomial coefficients

\[ \binom{n}{i} = \frac{n!}{i!(n-i)!} \]

The \( n! \)'s cancel, leaving

\[ e^{z+w} = \sum_{n,i} \frac{1}{i!(n-i)!} z^{n-i}w^i = e^z \cdot e^w \]

[1] Recall that the factorials \( n! \) for \( n = 1, 2, \ldots \) are \( n! = n(n-1)(n-2)\ldots3 \cdot 2 \cdot 1 \) and \( 0! = 1. \)
The second assertion is a consequence of the fact that the power series has real coefficients, and that complex conjugation is a \textit{continuous} map of the complex numbers to themselves. Thus, since the partial sums of $e^z$ conjugate as indicated, the limit does as well.

\[0.0.2\text{ Corollary: } (e^z)^n = e^{nz} \text{ for } n \in \mathbb{Z}.\]

\textbf{Proof:} For positive integers $n$, by induction

$$(e^z)^{n+1} = (e^z)^n \cdot e^z = e^{nz} \cdot e^z = e^{nz+z} = e^{(n+1)z}$$

Next, since

$$1 = e^0 = e^{z+(-z)} = e^z \cdot e^{-z}$$

we have $(e^z)^{-1} = e^{-z}$, and a similar induction gives $(e^z)^{-n} = e^{-nz}$ for negative integers $-n$. \[\big/\big/\]

\[0.0.3\text{ Corollary: } |e^{ix}| = 1 \text{ for real } x.\]

\textbf{Proof:} Simply

$$|e^{ix}|^2 = e^{ix} e^{ix} = e^{ix} e^{-ix} = e^0 = 1$$

using $x \in \mathbb{R}$. \[\big/\big/\]

We grant that the trigonometric functions $\sin x$ and $\cos x$ satisfy the differential equation $u'' = -u$ with boundary conditions $\sin' 0 = \cos 0 = 1 \quad \cos' 0 = \sin 0 = 0$

Again solving for \textit{power series} solutions, first assuming convergence, we obtain

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Then

$$e^{iz} = \cos z + i \sin z$$

for all real or complex $z$.

\[0.0.4\text{ Corollary: } \text{For real or complex } z\]

$$\cos^2 z + \sin^2 z = 1$$

\textbf{Proof:} We have

$$1 = e^{iz} e^{-iz} = (\cos z + i \sin z)(\cos z - i \sin z) = \cos^2 z + \sin^2 z$$

for any complex $z$. \[\big/\big/\]

\[0.0.5\text{ Lemma: } \text{The function } \cos x \text{ defined by the power series has \textit{least positive} zero between } \frac{4}{3} \text{ and } \frac{5}{3}.\]

\textbf{Proof:} From elementary estimates, noting that everything is real, for $0 \leq x \leq \frac{4}{3}$ the series for $\cos x$ is \textit{alternating decreasing} after the first two terms, so

$$\cos x \geq 1 - \frac{x^2}{2} \geq 1 - \left(\frac{4}{3}\right)^2 = 1 - \frac{8}{9} > 0$$

while

$$\cos \frac{5}{3} \leq 1 - \left(\frac{5}{2}\right)^2 \frac{4}{2!} + \left(\frac{5}{3}\right)^4 = 1 - \frac{25}{18} + \frac{625}{81 \cdot 24} = - \frac{131}{1944} < 0$$

By the intermediate value theorem, there is a $0$ between. \[\big/\big/\]

Declaring $\pi$ to be defined by saying that the first positive real 0 of $\cos x$ is $\pi/2$,

\[0.0.6\text{ Corollary: } e^{z+2\pi i} = e^z \quad e^{\pi i} = -1 \quad e^{\pi i/2} = i\]