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The exponential function, sine, cosine

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/03_exp_sin_cos.pdf]

We attempt to solve the differential equation (with initial condition)

$$u' = u \qquad u(0) = 1$$

in terms of a power series

$$u(x) = \sum_{n=0}^{\infty} c_n \, x^n$$

From Abel's theorem, assuming convergence,

$$u'(x) = \sum_{n=0}^{\infty} nc_n \, x^{n-1} \qquad u(0) = c_o$$

and the coefficients are uniquely determined by the values of the function. This implies a sequence of equalities

$$c_o = 1, \;\; 1 \cdot c_1 = c_o, \; 2 \cdot c_2 = c_1, \;\; 3 \cdot c_3 = c_2, \;\; \ldots$$

from which[1] $c_n = 1/n!$ and

$$u(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

This outcome was under an assumption that the series so-obtained is convergent, and, by the ratio test, it is indeed (absolutely) convergent for all real x, and, similarly, for all *complex* x. Write

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}$$

Note that it converges *uniformly* absolutely for z in compact subsets of the complex plane.

[0.0.1] Proposition: For complex z, w,

$$e^{z+w} = e^z \cdot e^w$$

The *complex conjugate* $\overline{e^z}$ of e^z is $e^{\bar{z}}$.

Proof: The first assertion is a corollary of the binomial theorem, as follows. With absolute convergence justifying rearrangements:

$$e^{z+w} = \sum_{n \geq 0} \frac{(z+w)^n}{n!} = \sum_{n,i} \binom{n}{i} z^{n-i} w^i / n!$$

with usual binomial coefficients

$$\binom{n}{i} = \frac{n!}{i! \, (n-i)!}$$

The $n!$'s cancel, leaving

$$e^{z+w} = \sum_{n,i} \frac{1}{i! \, (n-i)!} \, z^{n-i} w^i = e^z \cdot e^w$$

[1] Recall that the *factorials* $n!$ for $n = 1, 2, \ldots$ are $n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$ and $0! = 1$.

The second assertion is a consequence of the fact that the power series has real coefficients, and that complex conjugation is a *continuous* map of the complex numbers to themselves. Thus, since the partial sums of e^z conjugate as indicated, the limit does as well. ///

[0.0.2] Corollary: $(e^z)^n = e^{nz}$ for $n \in \mathbb{Z}$.

Proof: For positive integers n, by induction

$$(e^z)^{n+1} \;=\; (e^z)^n \cdot e^z \;=\; e^{nz} \cdot e^z \;=\; e^{nz+z} \;=\; e^{(n+1)z}$$

Next, since

$$1 \;=\; e^0 \;=\; e^{z+(-z)} \;=\; e^z \cdot e^{-z}$$

we have $(e^z)^{-1} = e^{-z}$, and a similar induction gives $(e^z)^{-n} = e^{-nz}$ for negative integers $-n$. ///

[0.0.3] Corollary: $|e^{ix}| = 1$ for real x.

Proof: Simply

$$|e^{ix}|^2 \;=\; e^{ix}\overline{e^{ix}} \;=\; e^{ix}e^{-ix} \;=\; e^0 \;=\; 1$$

using $x \in \mathbb{R}$. ///

We grant that the trigonometric functions $\sin x$ and $\cos x$ satisfy the differential equation $u'' = -u$ with boundary conditions

$$\sin' 0 = \cos 0 = 1 \qquad\qquad \cos' 0 = \sin 0 = 0$$

Again solving for *power series* solutions, first assuming convergence, we obtain

$$\sin x \;=\; \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad\qquad \cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Then

$$e^{iz} \;=\; \cos\, z + i \sin\, z$$

for all real or complex z.

[0.0.4] Corollary: For real or complex z

$$\cos^2 z + \sin^2 z \;=\; 1$$

Proof: We have

$$1 \;=\; e^{iz}e^{-iz} \;=\; (\cos\, z + i \sin\, z)(\cos\, z - i \sin\, z) \;=\; \cos^2 z + \sin^2 z$$

for any complex z. ///

[0.0.5] Lemma: The function $\cos\, x$ defined by the power series has *least positive zero* between $\frac{4}{3}$ and $\frac{5}{3}$.

Proof: From elementary estimates, noting that everything is *real*, for $0 \le x \le \frac{4}{3}$ the series for $\cos x$ is *alternating decreasing* after the first two terms, so

$$\cos x \;\ge\; 1 - \frac{x^2}{2} \;\ge\; 1 - \frac{\left(\frac{4}{3}\right)^2}{2!} \;=\; 1 - \frac{8}{9} \;>\; 0$$

while

$$\cos \frac{5}{3} \;\le\; 1 - \frac{\left(\frac{5}{3}\right)^2}{2!} + \frac{\left(\frac{5}{3}\right)^4}{4!} \;=\; 1 - \frac{25}{18} + \frac{625}{81 \cdot 24} \;=\; -\,\frac{131}{1944} \;<\; 0$$

By the intermediate value theorem, there is a 0 between. ///

Declaring π to be defined by saying that the first positive real 0 of $\cos x$ is $\pi/2$,

[0.0.6] Corollary:

$$e^{z+2\pi i} \;=\; e^z \qquad\qquad e^{\pi i} \;=\; -1 \qquad\qquad e^{\pi i/2} \;=\; i$$