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The keyhole/Hankel contour and $\zeta(-n)$

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http://www.math.umn.edu/~garrett/m/complex/notes_2014-15/04a_keyhole_and_zeta.pdf]

The contour-integration trick illustrated here appeared in one of Riemann's proofs of analytic continuation of $\zeta(s)$. It almost immediately proves that values of $\zeta(s)$ at non-positive integers are *rational*, and shows the connection to the Laurent coefficients of $1/(e^t - 1)$ at $t = 0$.

[1.1] **An integral representation of $\Gamma(s) \cdot \zeta(s)$** Although the integral representation of $\zeta(s)$ using a theta function is perhaps better in the long run, there is a more elementary one:

[1.1.1] **Claim:** For $\text{Re}(s) > 1$,

$$\Gamma(s) \cdot \zeta(s) = \int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t}$$

Proof: Expand a geometric series, exchange sum and integral, and change variables:

$$\begin{aligned} \int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t} &= \int_0^\infty \frac{t^s e^{-t}}{1 - e^{-t}} \frac{dt}{t} = \int_0^\infty t^s \left(\sum_{n \geq 1} e^{-nt} \right) \frac{dt}{t} = \sum_{n \geq 1} \int_0^\infty t^s e^{-nt} \frac{dt}{t} \\ &= \sum_{n \geq 1} \frac{1}{n^s} \int_0^\infty t^s e^{-t} \frac{dt}{t} = \Gamma(s) \cdot \sum_{n \geq 1} \frac{1}{n^s} = \Gamma(s) \cdot \zeta(s) \end{aligned}$$

as claimed. ///

[1.2] **Keyhole/Hankel contour** The *keyhole* or *Hankel* contour is a path from $+\infty$ inbound along the real line to $\varepsilon > 0$, counterclockwise around a circle of radius ε at 0, back to ε on the real line, and outbound back to $+\infty$ along the real line.

The usual elementary application is to evaluation of integrals similar to $\int_0^\infty \frac{t^s}{t^2+1} dt$, with $0 < \text{Re}(s) < 1$. In such an example, analytically continuing counterclockwise around 0 has no impact on the denominator, but, significantly, the numerator changes by a factor $e^{2\pi is}$, since

$$t^s = (|t| \cdot e^{i\theta})^s = |t|^s \cdot e^{i\theta s} \quad (\text{and } \theta \text{ goes from } 0 \text{ to } 2\pi)$$

We want the out-bound value of t^s to be real-valued for real s , so the inbound version of t^s must be actually be $t^s \cdot e^{-2\pi is}$. The absolute value of the integrand goes to 0 as $|t| \rightarrow 0$, so the integral over the small circle goes to 0 as $\varepsilon \rightarrow 0$, as do the integrals to and from 0, ε along the real line.

Thus, letting H_ε be the Hankel contour with circle of radius $\varepsilon > 0$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{H_\varepsilon} \frac{t^s}{t^2 + 1} dt &= \lim_{\varepsilon \rightarrow 0} \left(\int_{+\infty}^\varepsilon \frac{(t \cdot e^{-2\pi i})^s}{t^2 + 1} dt + (\text{integral over little circle}) + \int_\varepsilon^{+\infty} \frac{t^s}{t^2 + 1} dt \right) \\ &= (1 - e^{-2\pi is}) \int_0^\infty \frac{t^s}{t^2 + 1} dt \end{aligned}$$

In this elementary example, the trick is to further modify H_ε by *not* going all the way to $+\infty$ outbound, but stopping at $+R$ for large positive R , traversing clockwise a large circle of radius R back to the positive

real axis, and then inbound to ε . The integrals from R to and from $+\infty$ go to 0 as $R \rightarrow +\infty$, as does the integral over the large circle, since

$$|\text{integral over big circle}| \leq \text{length} \cdot \text{max value} \leq 2\pi R \cdot \frac{R^{\text{Re}(s)}}{R^2 - 1}$$

For each R, ε , this gives a path integral (clockwise) over a *closed* path. By *residues*, this picks up $-2\pi i$ times the sum of the residues inside the path. Thus, we discover that the integrals do not depend on the parameters $0 < \varepsilon < 1 < R$. Keeping track of the relevant versions of t^s ,

$$\begin{aligned} (1 - e^{-2\pi is}) \int_0^\infty \frac{t^s dt}{t^2 + 1} &= -2\pi i \cdot \left((\text{residue at } t = i) + (\text{residue at } t = -i) \right) \\ &= -2\pi i \cdot \left(\frac{e^{-\frac{3}{2}\pi is}}{i + i} + \frac{e^{-\frac{1}{2}\pi is}}{-i - i} \right) = -\pi \cdot (e^{-\frac{3}{2}\pi is} - e^{-\frac{1}{2}\pi is}) \end{aligned}$$

That is,

$$\int_0^\infty \frac{t^s dt}{t^2 + 1} = \pi \cdot \frac{e^{-\frac{1}{2}\pi is} - e^{-\frac{3}{2}\pi is}}{1 - e^{-2\pi is}} = \pi \cdot \frac{e^{\frac{1}{2}\pi is} - e^{-\frac{1}{2}\pi is}}{e^{\pi is} - e^{-\pi is}} = \pi \cdot (e^{\frac{1}{2}\pi is} + e^{-\frac{1}{2}\pi is}) = 2\pi \cos \frac{\pi s}{2}$$

This is a charming and useful device, but a different secondary trick is applied to $\zeta(s)$:

[1.3] Evaluation of $\zeta(-n)$ The first part of the Hankel contour discussion gives

$$\Gamma(s) \cdot \zeta(s) = \int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t} = \frac{1}{1 - e^{-2\pi i(s-1)}} \cdot \lim_{\varepsilon \rightarrow 0} \int_{H_\varepsilon} \frac{t^s}{e^t - 1} \frac{dt}{t} = \frac{1}{1 - e^{-2\pi is}} \cdot \lim_{\varepsilon \rightarrow 0} \int_{H_\varepsilon} \frac{t^s}{e^t - 1} \frac{dt}{t}$$

For $\zeta(s)$, rewrite this as

$$\zeta(s) = \frac{1}{\Gamma(s) \cdot (1 - e^{-2\pi is})} \cdot \lim_{\varepsilon \rightarrow 0} \int_{H_\varepsilon} \frac{t^s}{e^t - 1} \frac{dt}{t}$$

At $s = -n \in \{0, -1, -2, -4, \dots\}$ two fortunate things happen. First, the pole of $\Gamma(s)$ and the zero of $1 - e^{-2\pi is}$ cancel, giving a finite, computable value. Second, the function t^{-n-1} is *single-valued*, so the inbound and outbound integrals of the Hankel contour simply *cancel* each other, *and* the integral over the small circle at 0 becomes $2\pi i$ times the residue of $\frac{t^{-n-1}}{e^t - 1}$ at 0.

The periodicity of $1 - e^{2\pi is}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1 - e^{2\pi is} = 1 - \left(1 + \frac{-2\pi is}{1!} + \frac{(-2\pi is)^2}{2!} + \dots \right) = 2\pi is + \text{higher}$$

Grant for the moment that the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$\zeta(-n) = \frac{1}{\frac{(-1)^n}{n!} \cdot 2\pi i} \cdot 2\pi i \cdot \text{Res}_{t=0} \frac{t^{-n-1}}{e^t - 1} = (-1)^n \cdot n! \cdot \text{Res}_{t=0} \frac{t^{-n-1}}{e^t - 1}$$

The Laurent coefficients of $\frac{t^{-n-1}}{e^t - 1}$ are more-or-less Bernoulli numbers. These are not completely elementary objects, but are certainly *rational*. Thus, $\zeta(-n) \in \mathbb{Q}$.

[1.4] Vanishing $\zeta(-2) = \zeta(-4) = \dots = 0$ A slightly finer analysis of the generating function $\frac{1}{e^t - 1}$ yields the vanishing of $\zeta(s)$ at negative even integers, as follows.

First, $\frac{1}{e^t-1}$ is very close to being *odd* as a function of t :

$$\frac{1}{e^t-1} + \frac{1}{e^{-t}-1} = \frac{1}{e^t-1} + \frac{e^t}{1-e^t} = \frac{1}{e^t-1} - \frac{e^t}{e^t-1} = \frac{1-e^t}{e^t-1} = -1$$

Thus,

$$\left(\frac{1}{e^t-1} + \frac{1}{2}\right) + \left(\frac{1}{e^{-t}-1} + \frac{1}{2}\right) = 0$$

and $\frac{1}{e^t-1} + \frac{1}{2}$ is *odd*, so all its non-vanishing Laurent coefficients are odd-degree. Thus, for even $-2n < 0$,

$$\zeta(-2n) = (-1)^{2n}(2n)! \operatorname{Res}_{t=0} \frac{t^{-2n-1}}{e^t-1} = (2n)! (2n^{\text{th}} \text{ Laurent coefficient of } \frac{1}{e^t-1}) = 0$$

[1.5] Residues of $\Gamma(s)$ Finally, we determine the residues of $\Gamma(s)$. Certainly

$$\Gamma(1) = \int_0^\infty t^1 e^{-t} \frac{dt}{t} = \int_0^\infty e^{-t} dt = 1$$

From the functional equation $s\Gamma(s) = \Gamma(s+1)$, near $s = 0$

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{1 + \text{higher}}{s} = \frac{1}{s} + (\text{holomorphic at } s = 0)$$

Thus, the residue at 0 is 1. Iterating the functional equation,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{(s+1)s} = \frac{\Gamma(s+3)}{(s+2)(s+1)s} = \dots = \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\dots(s+2)(s+1)s}$$

Thus, the leading Laurent term at $s = -n$ is

$$\begin{aligned} \frac{1}{s+n} \cdot \frac{\Gamma(s+n+1)}{(s+n-1)\dots(s+2)(s+1)s} \Big|_{s=-n} &= \frac{1}{s+n} \cdot \frac{\Gamma(-n+n+1)}{(-n+n-1)\dots(-n+2)(-n+1)(-n)} \\ &= \frac{1}{s+n} \cdot \frac{1}{(-1)(-2)(-3)\dots(-n+2)(-n+1)(-n)} = \frac{1}{s+n} \cdot \frac{(-1)^n}{n!} \end{aligned}$$

That is, the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$ as claimed.

Bibliography

[Hankel 1863] H. Hankel, *Die Euler'schen Integrale bei unbeschränkter Variabilität des Argumentes*, Leopold Voss, Leipzig, 1863, 44 pp.

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