More basic results arising from Cauchy’s theorem

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

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1. Maximum modulus principle
2. Open mapping theorem
3. Rouché’s theorem

1. Maximum modulus principle

Recall that an open subset of a topological space, such as \( \mathbb{C} \), is connected if it cannot be expressed as a disjoint union of two non-empty subsets.

[1.0.1] Theorem: A non-constant \( f \) holomorphic on a non-empty, connected open set \( U \subset \mathbb{C} \), does not assume its maximum absolute value on \( U \).

Proof: One natural approach is to combine a hypothetical interior maximum of the absolute value with Cauchy’s formula expressing that interior value in terms of values on a circle enclosing it.

Given \( z_0 \in U \) and a neighborhood \( V \) of \( z_0 \), we show that there is \( z_1 \in V \) with \( |f(z_1)| > |f(z_0)| \). If not, then \( |f(z_1)| \leq |f(z_0)| \) for every \( z_1 \) on a small circle of radius \( r > 0 \) about \( z_0 \) fitting inside \( V \). Letting \( \gamma \) be that circle, traced counter-clockwise, Cauchy’s formula gives an inequality

\[
|f(z_0)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw \right| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{it})| \, \left| \frac{d}{dt} re^{it} \right| \, dt = \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{it})| \, dt
\]

Since \( f \) is continuous, if \( |f(re^{it})| < |f(z_0)| \) at any single \( t \), then \( |f(re^{it})| < |f(z_0)| \) for \( t' \) in a small-enough neighborhood of \( t \in \mathbb{R} \), and the inequality following from Cauchy’s formula would be impossible.

Thus, to avoid this contradiction, \( |f(z_1)| = |f(z_0)| \) for all \( z_1 \) on every sufficiently small circle near \( z_0 \). Thus, \( |f(z)| \) is constant, equal to \( |f(z_0)| \), near \( z_0 \).

Of course, if this constant absolute value is 0, then \( f \) is identically 0 on a neighborhood of \( z_0 \), so is identically 0 on the connected set \( U \), by the identity principle.

If the constant absolute value is not 0, then there is a holomorphic logarithm \( L \) defined on a sufficiently small neighborhood of \( f(z_0) \), and \( L(f(z)) \) is a holomorphic, purely-imaginary-valued function on a neighborhood of \( z_0 \). For \( z \) in such a small neighborhood of \( z_0 \),

\[
\lim_{h \to 0} \frac{L(f(z+h)) - L(f(z))}{h} = (L \circ f)'(z) = \lim_{h \to 0} \frac{L(f(z + ih)) - L(f(z))}{ih} = (L \circ f)'(z) \quad (h \in \mathbb{R})
\]

That is, the derivative is both real and purely imaginary, so is 0. Thus, \( L \circ f \) is constant. From this, as usual, by taking a derivative,

\[
0 = (L \circ f)'(z) = f'(z) \cdot L'(f(z)) = f'(z) \cdot \frac{1}{f(z)}
\]

giving \( f'(z) = 0 \). Thus, an interior maximum absolute value implies that \( f \) is constant.

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[1.0.2] Corollary: Let \( V \subset \mathbb{C} \) be a non-empty connected open with bounded closure \( \overline{V} \). The sup of non-constant holomorphic \( f \) on \( V \) extending continuously to \( \overline{V} \) occurs on the boundary \( \partial V \) of \( V \).
Proof: A continuous function on a compact set assumes its sup. Since \( f \) is non-constant, by the theorem this sup cannot occur in the \textit{interior} \( V \) of \( V \), so must occur on the boundary. ///

2. Open mapping theorem

[\textbf{2.0.1} Theorem:] A non-constant holomorphic function is an \textit{open} function, in the sense that it maps open sets to open sets.

\textbf{Proof:} This can be arranged as a corollary of the \textit{argument principle}.

Let \( f \) be holomorphic on a neighborhood \( U \) of \( z_o \), and let \( w_o = f(z_o) \), and where \( f(z) - w_o \) has a zero of multiplicity \( \mu \geq 1 \) at \( z_o \). We show that \( f(U) \) contains a neighborhood of \( w_o \), that is, that any \( w \) sufficiently near \( w_o \) is in \( f(U) \). To this end, consider an argument-principle integral which counts the number of zeros of \( f(z) - w_o \) inside a small simple closed curve \( \gamma \) around \( z_o \):

\[
\mu = \frac{1}{2\pi i} \int_{\gamma} d \log \left( f(z) - w_o \right) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z) \, dz}{f(z) - w_o}
\]

The function

\[
g(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z) \, dz}{f(z) - w}
\]

is \textit{holomorphic}, immediately from the definition of complex differentiability. At the same time, it is \textit{integer-valued}, by the argument principle, and takes value \( \mu \) at \( w_o \). Thus, \( g(w) \) is constant on a sufficiently small neighborhood of \( w_o \), so takes value \( \geq 1 \) on such a neighborhood. That is, every \( w \) in such a neighborhood is inside \( f(U) \). ///

3. Rouché’s theorem

[\textbf{3.0.1} Theorem:] Let \( f \) be holomorphic on an open set \( U \) containing a simple closed path \( \gamma \) and containing the interior of \( \gamma \). Suppose that \( f \) does not vanish on the path \( \gamma \). If another holomorphic function \( g \) on \( U \) satisfies

\[
|f(z) - g(z)| < |f(z)| \quad \text{(for all } z \text{ on } \gamma)\]

then the number of zeros of \( g \) inside \( \gamma \) is the same as the number of zeros of \( f \) inside \( \gamma \).

\textbf{Proof:} The function \( F = g/f \) is \textit{meromorphic} on \( U \) since the zeros of \( f \) are of finite order and cannot have an accumulation point in \( U \), by the identity principle. From the given inequality and from the non-vanishing of \( f \) on \( \gamma \),

\[
\left| 1 - \frac{g(z)}{f(z)} \right| < 1 \quad \text{(for } z \text{ on } \gamma)\]

That is, the values of \( F = g/f \) along \( \gamma \) stay inside the open disk \( D \) of radius 1 centered at 1. In particular, there is a holomorphic logarithm defined on \( D \), so by Cauchy’s theorem

\[
\int_{\gamma} \log F(z) \, dz = \int_{F \circ \gamma} \log w \, dw = 0
\]

On the other hand, by the argument principle,

\[
\left( \text{number of zeros of } F - \text{number of poles of } F \text{ inside } \gamma \right) = \frac{1}{2\pi i} \int_{\gamma} d \log F(z) = 0
\]
That difference is also
\[
\left( \text{number of zeros of } g - \text{number of zeros of } f \text{ inside } \gamma \right)
\]
even if some zeros of \( g \) cancel some zeros of \( f \) in the quotient \( F = g/f \). Thus, the number of zeros of \( g \) inside \( \gamma \) is the number of zeros of \( f \) there.

**3.0.2 Corollary: (Continuity of zeros)** Let \( f \) be a non-constant holomorphic function on an open set \( U \), \( h \) another holomorphic function on \( U \), and \( z_o \in U \) a simple zero of \( f \). Given \( \varepsilon > 0 \), for sufficiently small \( \delta > 0 \) there is a unique zero \( z_\delta \) of \( f + \delta h \) such that \( |z_o - z_\delta| < \varepsilon \).

**Proof:** Shrink \( \varepsilon > 0 \) if necessary so that \( f \) has no zeros on the circle of radius \( \varepsilon \) about \( z_o \). That circle is compact, so the continuous non-zero function \( z \to |f(z)| \) has a strictly positive minimum \( m \) there, and \( |h(z)| \) has a finite maximum \( M \) there. With \( 0 < \delta < \frac{m}{M} \),
\[
|f(z) - (f(z) + \delta h(z))| = \delta \cdot |h(z)| < \frac{m}{M} \cdot M \leq m |f(z)| \quad \text{(for } |z - z_o| = \varepsilon) \]
By Rouché’s theorem, \( f + \delta h \) has the same number of zeros inside \( |z - z_o| = \varepsilon \) as does \( f \), namely, a single one.