(*November 3, 2014*)

More basic results arising from Cauchy's theorem

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1. *Maximum modulus principle*

Recall that an open subset of a topological space, such as $\mathbb{C}$, is *connected* if it cannot be expressed as a disjoint union of two non-empty subsets.

[1.0.1] Theorem: A *non-constant* f holomorphic on a non-empty, connected open set $U \subset \mathbb{C}$, does *not* assume its maximum absolute value on U.

Proof: One natural approach is to combine a hypothetical *interior* maximum of the absolute value with Cauchy's formula expressing that interior value in terms of values on a circle enclosing it.

Given $z_o \in U$ and a neighborhood V of z_o, we show that there is $z_1 \in V$ with $|f(z_1)| > |f(z_o)|$. If not, then $|f(z_1)| \le |f(z_o)|$ for every z_1 on a small circle of radius $r > 0$ about z_o fitting inside V. Letting γ be that circle, traced counter-clockwise, Cauchy's formula gives an inequality

$$|f(z_o)| \;=\; \Big|\frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w-z}\Big| \;\le\; \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(re^{it})|\,\big|\frac{d}{dt}re^{it}\big|\,dt}{r} \;=\; \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|\,dt$$

Since f is continuous, if $|f(re^{it}| < |f(z_o)|$ at any single t, then $|f(re^{it'}| < |f(z_o)|$ for t' in a small-enough neighborhood of $t \in \mathbb{R}$, and the inequality following from Cauchy's formula would be impossible.

Thus, to avoid this contradiction, $|f(z_1)| = |f(z_o)|$ for all z_1 on every sufficiently small circle near z_o. Thus, $|f(z)|$ is *constant*, equal to $|f(z_o)|$, near z_o.

Of course, if this constant absolute value is 0, then f is identically 0 on a neighborhood of z_o, so is identically 0 on the connected set U, by the identity principle.

If the constant absolute value is *not* 0, then there is a holomorphic logarithm L defined on a sufficiently small neighborhood of $f(z_o)$, and $L(f(z))$ is a holomorphic, purely-imaginary-valued function on a neighborhood of z_o. For z in such a small neighborhood of z_o,

$$\lim_{h\to 0}\frac{L(f(z+h)) - L(f(z))}{h} \;=\; (L\circ f)'(z) \;=\; \lim_{h\to 0}\frac{L(f(z+ih)) - L(f(z))}{ih} \;=\; (L\circ f)'(z) \qquad (h \in \mathbb{R})$$

That is, the derivative is both real and purely imaginary, so is 0. Thus, $L \circ f$ is *constant*. From this, as usual, by taking a derivative,

$$0 \;=\; (L\circ f)'(z) \;=\; f'(z)\cdot L'(f(z)) \;=\; f'(z)\cdot\frac{1}{f(z)}$$

giving $f'(z) = 0$. Thus, an interior maximum absolute value implies that f is constant. ///

[1.0.2] Corollary: Let $V \subset \mathbb{C}$ be a non-empty connected open with *bounded* closure $\overline{V}$. The sup of non-constant holomorphic f on V extending continuously to $\overline{V}$ occurs on the *boundary* ∂V of V.

Proof: A continuous function on a compact set assumes its sup. Since f is non-constant, by the theorem this sup cannot occur in the *interior* V of $\overline{V}$, so must occur on the boundary. ///

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2. *Open mapping theorem*

[2.0.1] Theorem: A non-constant holomorphic function is an *open* function, in the sense that it maps open sets to open sets.

Proof: This can be arranged as a corollary of the *argument principle*.

Let f be holomorphic on a neighborhood U of z_o, and let $w_o = f(z_o)$, and where $f(z) - w_o$ has a zero of multiplicity $\mu \geq 1$ at z_o. We show that $f(U)$ contains a neighborhood of w_o, that is, that any w sufficiently near w_o is in $f(U)$. To this end, consider an argument-principle integral which counts the number of zeros of $f(z) - w_o$ inside a small simple closed curve γ around z_o:

$$\mu \;=\; \frac{1}{2\pi i}\int_\gamma d\big(\log\big(f(z) - w_o\big)\big) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - w_o}$$

The function

$$g(w) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - w}$$

is *holomorphic*, immediately from the definition of complex differentiability. At the same time, it is *integer-valued*, by the argument principle, and takes value μ at w_o. Thus, $g(w)$ is constant on a sufficiently small neighborhood of w_o, so takes value ≥ 1 on such a neighborhood. That is, every w in such a neighborhood is inside $f(U)$. ///

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3. *Rouché's theorem*

[3.0.1] Theorem: Let f be holomorphic on an open set U containing a simple closed path γ and containing the interior of γ. Suppose that f does not vanish on the path γ. If another holomorphic function g on U satisfies

$$|f(z) - g(z)| \;<\; |f(z)| \qquad\qquad \text{(for all } z \text{ on } \gamma)$$

then the number of zeros of g inside γ is the same as the number of zeros of f inside γ.

Proof: The function $F = g/f$ is *meromorphic* on U since the zeros of f are of finite order and cannot have an accumulation point in U, by the identity principle. From the given inequality and from the non-vanishing of f on γ,

$$\Big|1 - \frac{g(z)}{f(z)}\Big| \;<\; 1 \qquad\qquad \text{(for } z \text{ on } \gamma)$$

That is, the values of $F = g/f$ along γ stay inside the open disk D of radius 1 centered at 1. In particular, there is a holomorphic logarithm defined on D, so by Cauchy's theorem

$$\int_\gamma \log F(z)\,dz \;=\; \int_{F\circ\gamma} \log w\,dw \;=\; 0$$

On the other hand, by the argument principle,

$$\Big(\text{number of zeros of } F \text{ - number of poles of } F \text{ inside } \gamma\Big) \;=\; \frac{1}{2\pi i}\int_\gamma d(\log F(z)) \;=\; 0$$

That difference is also

$$\Big(\text{number of zeros of } g \text{ - number of zeros of } f \text{ inside } \gamma\Big)$$

even if some zeros of g cancel some zeros of f in the quotient $F = g/f$. Thus, the number of zeros of g inside γ is the number of zeros of f there. ///

[3.0.2] Corollary: *(Continuity of zeros)* Let f be a non-constant holomorphic function on an open set U, h another holomorphic function on U, and $z_o \in U$ a *simple* zero of f. Given $\varepsilon > 0$, for sufficiently small $\delta > 0$ there is a unique zero z_δ of $f + \delta h$ such that $|z_o - z_\delta| < \varepsilon$.

Proof: Shrink $\varepsilon > 0$ if necessary so that f has no zeros on the circle of radius ε about z_o. That circle is *compact*, so the continuous non-zero function $z \to |f(z)|$ has a strictly positive minimum m there, and $|h(z)|$ has a finite maximum M there. With $0 < \delta < \frac{m}{M}$,

$$\Big|f(z) - \big(f(z) + \delta h(z)\big)\Big| \;=\; \delta \cdot |h(z)| \;<\; \frac{m}{M} \cdot M \;\leq\; m\,|f(z)| \qquad\qquad (\text{for } |z - z_o| = \varepsilon)$$

By Rouché's theorem, $f + \delta h$ has the same number of zeros inside $|z - z_o| = \varepsilon$ as does f, namely, a single one. ///
