**Phragmén-Lindelöf Theorems**

Paul Garrett  
garrett@math.umn.edu  
http://www.math.umn.edu/~garrett/

The paper that gave its name to these results is E. Phragmén, E. Lindelöf, *Sur une extension d’un principe classique de l’analyse*, Acta Math. 31 (1908), 381-406 proved the theorem here.

The maximum modulus principle can easily be misapplied on unbounded open sets. That is, while for an open set \( U \subset \mathbb{C} \) with bounded closure \( \overline{U} \), it does follow that the sup of a holomorphic function \( f \) on \( U \) extending continuously to \( \overline{U} \) occurs on the boundary \( \partial U \) of \( U \), holomorphic functions on an unbounded set can be bounded by 1 on the edges but be violently unbounded in the interior.

A simple example is \( f(z) = e^{e^{x+y}} \):

\[
|e^{e^{x+y}}| = e^{e^{x} \cos y}
\]

On one hand, for fixed \( y = \text{Im} \ z \) with \( \cos y > 0 \), the function blows up as \( x = \text{Re} \ z \to +\infty \). On the other hand, for \( \cos y = 0 \) the function is bounded. Thus, on the strip \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \), the function \( e^{e^{z}} \) is bounded on the edges but blows up as \( x \to +\infty \).

This example suggests growth conditions under which a bound of 1 on the edges implies the same bound throughout the strip. In fact, the suggested bound is essentially sharp.

**[0.0.1] Theorem:** For \( f \) a holomorphic function on the horizontal half-strip

\[
\{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } 0 \leq x\}
\]

satisfying

\[
|f(z)| \ll e^{C \text{Re} z} \quad \text{(for some constant } 0 \leq C < 1 \text{)}
\]

\(|f(z)| \leq 1\) on the edges of the half-strip implies \(|f(z)| \leq 1\) in the interior, as well.

**Proof:** Unsurprisingly, the proof is a reduction to the usual maximum modulus principle. Take any fixed \( D \) in the range

\[
C < D < 1
\]

The function

\[
F_\varepsilon(z) = f(z)/e^{\varepsilon e^{D \cdot x}} \quad \text{(for } \varepsilon > 0 \text{)}
\]

is bounded by 1 on the edges of the half-strip, and in the interior goes to 0 uniformly in \( y \) as \( x \to +\infty \), for fixed \( \varepsilon > 0 \), exploiting the modification with \( D \). Thus, on a rectangle

\[
R_T = \{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } 0 \leq x \leq T\}
\]

for sufficiently large \( T > 0 \) depending upon \( \varepsilon \), the function \( F_\varepsilon \) is bounded by 1 on the edge. The usual maximum modulus principle implies that \( F_\varepsilon \) is bounded by 1 throughout. That is, for each fixed \( z_0 \) in the half-strip,

\[
|f(z_0)| \leq e^{\varepsilon e^{D \text{Re} z_0}} \quad \text{(for all } \varepsilon > 0 \text{)}
\]

Let \( \varepsilon \to 0^+ \), giving \(|f(z_0)| \leq 1\).  

**[0.0.2] Remark:** Analogous theorems on strips of other widths follow by using \( e^{ce^z} \) with suitable constants \( c \).
An analogous theorem on a full strip, rather than half-strip, follows by using a function like \( e^{\cosh z} \) in place of \( e^{z} \), as follows.

**[0.0.3] Theorem:** For \( f \) a holomorphic function on the full horizontal strip

\[
\{ z : -\frac{\pi}{2} \leq \Im z \leq \frac{\pi}{2} \}
\]
satisfying

\[
|f(z)| \ll e^{C \cosh \Re z} \quad \text{(for some constant } 0 \leq C < 1)\]

\( |f(z)| \leq 1 \) on the edges of the strip implies \( |f(z)| \leq 1 \) in the interior, as well.

**Proof:** Again, reduce to the maximum modulus principle. Fix \( D \) in the range \( C < D < 1 \). The function

\[
F_\varepsilon(z) = \frac{f(z)}{e^{\varepsilon \cosh D z}} \quad \text{(for }\varepsilon > 0)\]

is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in \( y \) as \( x \to \pm \infty \), for fixed \( \varepsilon > 0 \). Thus, on a rectangle

\[
R_T = \{ z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } -T \leq x \leq T \} \quad \text{(for large } T > 0, \text{ depending upon } \varepsilon)\]

the function \( F_\varepsilon \) is bounded by 1 on the edge. The usual maximum modulus principle implies that \( F_\varepsilon \) is bounded by 1 throughout. That is, for each fixed \( z_o \) in the half-strip,

\[
|f(z_o)| \leq e^{\varepsilon \cosh D \Re z_o} \quad \text{(for all } \varepsilon > 0)\]

We can let \( \varepsilon \to 0^+ \), giving \( |f(z_o)| \leq 1 \). \( /// \)

The details of various adjustments can be made to disappear by strengthening the hypotheses:

**[0.0.4] Corollary:** Let \( f \) be a holomorphic function on a strip or half-strip, with a bound

\[
|f(z)| \ll e^{A|z|} \quad \text{(for some } A > 0)\]

If \( |f(z)| \leq 1 \) on the edges of the (half-)strip, then \( |f(z)| \leq 1 \) in the interior, as well. \( /// \)

**[0.0.5] Remark:** Further variations are easily possible, by additional adjustments of functions. For example, polynomial growth of a function \( f \) on the edges of a strip or half-strip can be accommodated by considering \( f(z)/(z - z_o)^M \) for \( z_o \) outside the strip, and large \( M \).