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Phragmén-Lindelöf Theorems

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The paper that gave its name to these results is

E. Phragmén, E. Lindelöf, *Sur une extension d'un principe classique de l'analyse*, Acta Math. **31** (1908), 381-406 proved the theorem here.

The *maximum modulus principle* can easily be misapplied on *unbounded* open sets. That is, while for an open set $U \subset \mathbb{C}$ with *bounded* closure $\overline{U}$, it *does* follow that the sup of a holomorphic function f on U extending continuously to $\overline{U}$ occurs on the boundary ∂U of U, holomorphic functions on an *unbounded* set can be bounded by 1 on the edges but be violently unbounded in the interior.

A simple example is $f(z) = e^{e^z}$:

$$\left| e^{e^{x+iy}} \right| \;=\; e^{\mathrm{Re}\,(e^{x+iy})} \;=\; e^{e^x \cdot \cos y}$$

On one hand, for fixed $y = \mathrm{Im}\, z$ with $\cos\, y > 0$, the function blows up as $x = \mathrm{Re}\, z \to +\infty$. On the other hand, for $\cos y = 0$ the function is *bounded.* Thus, on the strip $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, the function e^{e^z} is bounded on the edges but blows up as $x \to +\infty$.

This example suggests *growth conditions* under which a bound of 1 on the edges implies the same bound throughout the strip. In fact, the suggested bound is essentially sharp.

[0.0.1] Theorem: For f a holomorphic function on the horizontal half-strip

$$\{z \;:\; -\frac{\pi}{2} \;\le\; y \;\le\; \frac{\pi}{2} \;\text{ and }\; 0 \;\le\; x\}$$

satisfying

$$|f(z)| \;\ll\; e^{e^{C\cdot \mathrm{Re}\, z}} \qquad \text{(for some constant } 0 \le C < 1)$$

$|f(z)| \le 1$ on the edges of the half-strip implies $|f(z)| \le 1$ in the interior, as well.

Proof: Unsurprisingly, the proof is a reduction to the usual maximum modulus principle. Take any fixed D in the range

$$C \;<\; D \;<\; 1$$

The function

$$F_\varepsilon(z) \;=\; f(z)/e^{\varepsilon e^{D\cdot z}} \qquad (\text{for } \varepsilon > 0)$$

is bounded by 1 on the edges of the half-strip, and in the interior goes to 0 uniformly in y as $x \to +\infty$, for fixed $\varepsilon > 0$, exploiting the modification with D. Thus, on a rectangle

$$R_T \;=\; \{z \;:\; -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ and } 0 \le x \le T\}$$

for sufficiently large $T > 0$ depending upon ε, the function F_ε is bounded by 1 on the edge. The usual maximum modulus principle implies that F_ε is bounded by 1 throughout. That is, *for each fixed* z_o in the half-strip,

$$|f(z_o)| \;\le\; e^{\varepsilon\cdot e^{D\mathrm{Re}\, z_o}} \qquad (\text{for all } \varepsilon > 0)$$

Let $\varepsilon \to 0^+$, giving $|f(z_o)| \le 1$. ///

[0.0.2] Remark: Analogous theorems on strips of other widths follow by using $e^{c\cdot e^z}$ with suitable constants c.

An analogous theorem on a full strip, rather than half-strip, follows by using a function like $e^{\cosh z}$ in place of e^{e^z}, as follows.

[0.0.3] Theorem: For f a holomorphic function on the full horizontal strip

$$\{z \;:\; -\frac{\pi}{2} \;\le\; \mathrm{Im}\, z \;\le\; \frac{\pi}{2}\}$$

satisfying

$$|f(z)| \;\ll\; e^{\cosh C\cdot \mathrm{Re}\, z} \qquad \text{(for some constant } 0 \le C < 1)$$

$|f(z)| \le 1$ on the edges of the strip implies $|f(z)| \le 1$ in the interior, as well.

Proof: Again, reduce to the maximum modulus principle. Fix D in the range $C < D < 1$. The function

$$F_\varepsilon(z) \;=\; f(z)/e^{\varepsilon \cosh Dz} \qquad \text{(for } \varepsilon > 0)$$

is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in y as $x \to \pm\infty$, for fixed $\varepsilon > 0$. Thus, on a rectangle

$$R_T \;=\; \{z \;:\; -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ and } -T \le x \le T\} \qquad \text{(for large } T > 0\text{, depending upon } \varepsilon)$$

the function F_ε is bounded by 1 on the edge. The usual maximum modulus principle implies that F_ε is bounded by 1 throughout. That is, *for each fixed* z_o in the half-strip,

$$|f(z_o)| \;\le\; e^{\varepsilon \,\cosh D \mathrm{Re}\; z_o} \qquad \text{(for all } \varepsilon > 0)$$

We can let $\varepsilon \to 0^+$, giving $|f(z_o)| \le 1$. ///

The details of various adjustments can be made to disappear by strengthening the hypotheses:

[0.0.4] Corollary: Let f be a holomorphic function on a strip or half-strip, with a bound

$$|f(z)| \;\ll\; e^{|z|^A} \qquad \text{(for some } A > 0)$$

If $|f(z)| \le 1$ on the edges of the (half-)strip, then $|f(z)| \le 1$ in the interior, as well. ///

[0.0.5] Remark: Further variations are easily possible, by additional adjustments of functions. For example, *polynomial growth* of a function f on the edges of a strip or half-strip can be accommodated by considering $f(z)/(z - z_o)^M$ for z_o outside the strip, and large M.
