Conformal mapping

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1. Conformal (angle-preserving) maps

A complex-valued function \( f \) on a non-empty open \( U \subset \mathbb{C} \) is conformal if it preserves angles, in the sense that, for any two smooth parametrized curves \( \gamma : [a,b] \to \mathbb{C}, \delta : [c,d] \to \mathbb{C} \) with \( \gamma(a) = z_0 = \delta(c) \), the angle between \( \gamma'(a) \) and \( \delta'(c) \) is equal to the angle between \( (f \circ \gamma)'(a) \) and \( (f \circ \delta)'(c) \). The function \( f \) is orientation-preserving if the directed angle is preserved.

Explicitly, for non-zero \( z, w \in \mathbb{C} \),

\[
\text{(directed angle from } z \text{ to } w) = \theta \in \mathbb{R}, \text{ such that } \frac{w}{|w|} = e^{i\theta} \frac{z}{|z|}
\]

That is,

\[
e^{i\theta} = \frac{w}{|w|} \bigg/ \frac{z}{|z|}
\]

[1.0.1] Claim: A holomorphic function \( f \) is conformal and orientation-preserving at points \( z_0 \) where \( f'(z_0) \neq 0 \).

Proof: This is a direct computation, using the chain rule, noting that \( f' \) is the complex derivative of \( f \), while \( \gamma' \) is the real derivative. With \( \gamma(a) = z_0 = \delta(c) \),

\[
(f \circ \gamma)'(a) = f'(\gamma(a)) \cdot \gamma'(a) = f'(z_0) \cdot \gamma'(a) \quad (f \circ \delta)'(c) = f'(\delta(a)) \cdot \delta'(a) = f'(z_0) \cdot \delta'(a)
\]

so

\[
\frac{\left| f'(\gamma(a)) \right| \left| (f \circ \gamma)'(a) \right|}{\left| (f \circ \delta)'(a) \right|} = \frac{\left| f'(z_0) \gamma'(a) \right| \left| f'(z_0) \delta'(a) \right|}{\left| \gamma'(a) \right| \left| \delta'(a) \right|}
\]

showing that directed angles are preserved. //

[1.0.2] Remark: Holomorphic \( f \) on \( U \) with non-vanishing derivative maps the mutually orthogonal families of lines \( \text{Re}(z) = x = \text{const} \) and \( \text{Im}(z) = y = \text{const} \) to mutually orthogonal curves.

For example, letting \( f(z) = z^2 \) on the upper half-plane, the lines \( x \to x + iy_0 \) become parabolas \( x \to x^2 - 2ixy_0 - y_0^2 \) opening horizontally, and the lines \( y \to x_0 + iy \) become parabolas \( y \to -y^2 + 2x_0y + x_0^2 \) opening vertically.

Letting \( f(z) = \sqrt{z} \) on \( \mathbb{C} - [0, +\infty) \) amounts to looking at inverse images in the previous example, giving mutually orthogonal lines \( \text{Re}(z^2) = \text{const} \) and \( \text{Im}(z^2) = \text{const} \), namely, hyperbolas \( x^2 - y^2 = \text{const} \) and hyperbolas \( xy = \text{const} \).
2. Lines and circles and linear fractional transformations

[2.0.1] Theorem: The collection of lines and circles in $\mathbb{C} \cup \{\infty\}$ is stabilized by linear fractional transformations, and is acted upon transitively by them.

Proof: Clearly affine maps $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) = (az + b)/d$, with $ad \neq 0$, which are combinations of translations, rotations, and dilations, preserve lines and circles. Given this, a group-theoretic result greatly simplifies things:

[2.0.2] Claim: (Bruhat decomposition) Let $P$ be the upper-triangular matrices in $GL_2(\mathbb{C})$, and the Weyl element $w_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$GL_2(\mathbb{C}) = P \sqcup Pw_oP \quad \text{(disjoint union)}$$

Proof: A group element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $P$ if $c = 0$, so consider $c \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in P$$

That is, for $g \notin P$, $gPw_o \cap P \neq \emptyset$. That is, $gP \cap Pw_o \neq \emptyset$, so $gP \cap Pw_oP \neq \emptyset$, and $g \in Pw_oP$. ///

Invoking the Bruhat decomposition, since $g \in P$ preserves lines and circles, it suffices to prove that the Weyl element $w_o$ preserves lines and circles. The real and imaginary parts of $w_o(z) = 1/z$ are easily observed:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

so

$$\text{Re}(1/z) = \frac{x}{|z|^2} \quad \text{Im}(1/z) = -\frac{y}{|z|^2}$$

It is convenient that $w_o^2$ gives the identity map, so the images and inverse images of sets under $w_o$ are the same things. Given a line $ax + by = c$ with real $a, b, c$, the image of the line is

$$a \frac{x}{|z|^2} - b \frac{y}{|z|^2} = c$$

or

$$ax - by = cx^2 + cy^2$$

giving the circle

$$(x - \frac{a}{c})^2 + (y + \frac{b}{c})^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$

The image of a circle $|z - z_o| = r$ is

$$\left|\frac{1}{z} - z_o\right| = r$$

or $|1 - z_o \cdot z| = r \cdot |z|$ or

$$1 - 2 \text{Re}(z_o \cdot z) + |z_o|^2 \cdot |z|^2 = r^2 \cdot |z|^2$$

In the special case that $|z_o| = r$, this is a line. Otherwise, it is a circle.
Three points on a circle determine the circle completely. A line in \( \mathbb{C} \) can be viewed determined by two points on it in \( \mathbb{C} \), and inevitably passing through \( \infty \). The group of linear fractional transformation actions is \textit{triply transitive} in the sense that it can map any triple of distinct points to any other, so is transitive on circles-and-lines.

\[
\begin{align*}
\end{align*}
\]

\[ \textbf{2.0.3 \ Remark:} \quad \text{Beware: except for \textit{affine} maps }
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}
\begin{pmatrix} z \\ 1 \end{pmatrix} = \frac{(az + b)}{d}, \text{ with } ad \neq 0, \text{ linear fractional transformations do } \textit{not} \ \text{respect centers of circles.} \]

\section{3. Elementary examples}

A \textit{sector} is an open set of the form
\[
\{ z = r e^{i \theta} : r > 0, \ a < \theta < b, \ \text{with } |b - a| < 2\pi \}
\]

Two sectors of the special form
\[
\begin{align*}
\{ z = r e^{i \theta} : r > 0, \ 0 < \theta < b, \ \text{with } b < 2\pi \} & \quad \{ z = r e^{i \theta} : r > 0, \ 0 < \theta < b', \ \text{with } b' < 2\pi \}
\end{align*}
\]
can be mapped holomorphically to each other by a \textit{power map}
\[
\begin{align*}
z & \rightarrow z^{b'/b} = e^{b'/b \log z}
\end{align*}
\]
with a continuous logarithm defined on the first sector by
\[
\begin{align*}
\log z & = i \pi + \int_1^{z/e^{i \pi}} \frac{dw}{w}
\end{align*}
\]
integrating along a straight line segment. In polar coordinates, this is simply
\[
\begin{align*}
re^{i \theta} & \rightarrow r^{b'/b} e^{i \theta^{b'/b}}
\end{align*}
\]
Exhibiting the map as a \textit{holomorphic} map shows that it preserves angles.

Sectors with edges elsewhere than the positive real axis can be rotated, by map \( z \rightarrow \mu \cdot z \) with |\( \mu \)| = 1, to put either edge on the positive real axis. Thus, the problem of mapping one sector to another reduces to that simpler case, by pre-composing and post-composing with rotations:
\[
\{ z = r e^{i \theta} : r > 0, \ a < \theta < b, \ \text{with } |b - a| < 2\pi \}
\]
can be mapped to another sector
\[
\{ z = r e^{i \theta} : r > 0, \ a' < \theta < b', \ \text{with } |b' - a'| < 2\pi \}
\]
by
\[
\begin{align*}
3 & \rightarrow \frac{z}{e^{i \alpha}} \rightarrow \left( \frac{z}{e^{i \alpha}} \right)^{b'-a'/b-a} \rightarrow e^{i \alpha'} \cdot \left( \frac{z}{e^{i \alpha}} \right)^{b'-a'/b-a}
\end{align*}
\]
Again, there is an unambiguous choice of continuous \( \alpha^{th} \) power on such a sector, by
\[
\begin{align*}
z^\alpha & = e^{\alpha \log z}
\end{align*}
\]
where \( \log z \) is defined on \( \mathbb{C} \) with any ray \( \{re^{i\theta} : r > 0 \} \) removed, with this ray not lying in the given sector. Again, such a logarithm can be defined by

\[
\log z = i(\theta_0 + \pi) + \int_1^{z/e^{i(\theta_0+\pi)}} \frac{dw}{w}
\]

integrating along a straight line segment from 1 to \( z/e^{i(\theta_0+\pi)} \). That is, all sectors are conformally equivalent.

In other words, while \( z \to z^\alpha \) with real \( \alpha \neq 0 \) is conformal at every point other than \( z = 0 \), it alters angles at 0 by multiplying angles by \( \alpha \).

A bigon is an open subset of \( \mathbb{C}\mathbb{P}^1 \) bounded by two arcs, by which we for present purposes we mean either straight line segments, possibly infinite in one or both directions, and/or arcs of circles with angle measure \( < 2\pi \).

The non-degenerate cases are specified by two vertices \( z_1, z_2 \in \mathbb{C}\mathbb{P}^1 \), and two distinct circles-or-lines passing through both \( z_1, z_2 \). This configuration cuts \( \mathbb{C}\mathbb{P}^1 \) into four connected components, each of which is a bigon.

In the degenerate case that the two points are identical, \( \mathbb{C}\mathbb{P}^1 \) is cut into only three connected components.

[3.0.1] Claim: All bigons in the non-degenerate case are conformally equivalent, and are conformally equivalent to the upper half-plane.

Proof: Given vertices \( z_1 \neq z_2 \), map \( z_1 \to 0 \) and \( z_2 \to \infty \) by a linear fractional transformation. Since linear fractional transformations preserve lines and circles, the boundary of the image consists of two straight lines from 0 to \( \infty \), so the image is a sector. Rotate the sector until is is of the form \( \{re^{i\theta} : r > 0, 0 < \theta < b, \text{ with } b < 2\pi \} \), and the apply a power map to obtain the upper half-plane \( \mathbb{H} \). Thus, any non-degenerate bigon is conformally equivalent to the upper half-plane, so they are equivalent to each other. ///

[3.0.2] Claim: All degenerate bigons are conformally equivalent, and are conformally equivalent to the strip \( 0 < \text{Re}(z) < 1 \).

Proof: Map the single vertex to \( \infty \). The two bounded arcs, mapped to arcs passing through \( \infty \), must become straight lines with no other intersections, thus, parallel. Translate, rotate and dilate to obtain the lines \( \text{Re}(z) = 0 \) and \( \text{Re}(z) = 1 \). ///

[3.0.3] Remark: An open disk can be considered as a non-degenerate bigon in many ways, as can a half-plane. The Cayley map

\[
z \mapsto \frac{z + i}{iz + 1}
\]

maps \( i \to \infty, -i \to 0, \) and \( 1 \to 1 \), so maps the unit circle to the real line, because linear fractional transformations preserve circles-and-lines. Thus, it maps the connected components of the complement of the circle to the connected components of the complement of the real line: since \( 0 \to i \), the interior of the circle is mapped to the upper half-plane.
4. $f'(z) = 0$ implies local non-injectivity

The following generally-useful corollary of the argument principle could have been proven earlier, but is perhaps of interest here as a sort of converse to the holomorphic inverse-function theorem:

[4.0.1] Claim: A non-constant holomorphic function $f$ near $z_0$ vanishing to order exactly $k$ at $z_0$ is $(k+1)$-to-1 in every sufficiently small punctured disk at $z_0$, in the following precise sense. Given a sufficiently small $r > 0$, there is a neighborhood $U$ of $z_0$ such that, for all $z_1$ in $U$, the value $f(z_1)$ is hit $k+1$ times inside $|z - z_0| = r$.

Proof: The idea is that, on one hand, since $f(z) - f(z_0)$ vanishes to order $k$ at $z_0$, the argument principle would give

$$k + 1 \leq \frac{1}{2\pi i} \int_\gamma \frac{(f(w) - f(z_0))'}{f(w) - f(z_0)} \, dw = \frac{1}{2\pi i} \int_\gamma \frac{f'(w) \, dw}{f(w) - f(z_0)},$$

where $\gamma$ is the counterclockwise path around $|z - z_0| = r$. On the other hand,

$$z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{f'(w) \, dw}{f(w) - f(z)}$$

should be a holomorphic function of $z$ on some neighborhood $U$ of $z_0$. An integer-valued holomorphic function is constant. Thus, it should be every value $f(z_1)$ for $z_1 \in U$ is hit at least $k + 1$ times inside the circle $|z - z_0| = r$.

Two details are missing. First, we want to be sure that the denominator $f(w) - f(z_0)$ does not vanish on the circle $|z - z_0| = r$. We claim that we can shrink $r$ if necessary so that $f(w) \neq f(z_0)$ on that circle: indeed, if there were a sequence of points $z_1, z_2, \ldots$ approaching $z_0$ with $f(z_j) = f(z_0)$, then by the identity principle $f$ is constant. Second, we want the denominator $f(w) - f(z)$ to not vanish for all $w$ on the circle and for all $z$ sufficiently near to $z_0$. Indeed, the set of images $I = \{f(z) : |z - z_0| = r\}$ is a continuous image of a compact set, so is compact, so is closed. Since $f(z_0) \notin I$, there is some neighborhood $N$ of $f(z_0)$ disjoint from $I$. The inverse image $f^{-1}(N)$ is open, by continuity, and contains $z_0$, so there is a neighborhood $U$ of $z_0$ such that $f(U) \cap I = \emptyset$. Shrink $U$ to be an open disk at $z_0$, so it is connected. ///

5. Automorphisms of the disk and of $\mathcal{H}$

First, we demonstrate some explicit groups of linear transformations stabilizing the upper half-plane, or stabilizing the unit disk, in both cases large enough to act transitively. Then we invoke Schwarz’ lemma (from the following section) to see that these groups are all the holomorphic automorphisms of these regions.

[5.0.1] Claim: The linear fractional transformations arising from

$$SL_2(\mathbb{R}) = \{\text{two-by-two real matrices with determinant } = 1\}$$

stabilize the upper half-plane $\mathcal{H}$, and act transitively on it. In particular, $\text{Im} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(z) = \frac{\text{Im}(z)}{|cz + d|^2}$.

Proof: First, $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$, including inverses. Directly compute the effect on imaginary parts:

$$2\text{Im} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(x + iy) = 2\text{Im} \frac{az + b}{cz + d} = \frac{az + b}{cz + d} - \frac{a\overline{z} + b}{\overline{cz + d}} = \frac{(az + b)(cz + d) - (a\overline{z} + b)(cz + d)}{|cz + d|^2} = \frac{(ad - bc)(z - \overline{z})}{|cz + d|^2} = \frac{z - \overline{z}}{|cz + d|^2}.$$
This shows that $SL_2(\mathbb{R})$ stabilizes the upper half-plane. To show transitivity, observe that for $x + iy \in \mathcal{H}$

$$
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\sqrt{y} & 0 \\
0 & \frac{1}{\sqrt{y}}
\end{pmatrix}
(i) = x + iy
$$

so the point $i \in \mathcal{H}$ can be mapped to any other.  

///

[5.0.2] Remark: The special orthogonal group

$$
SO_2(\mathbb{R}) = \{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \} \subset SL_2(\mathbb{R})
$$

is the stabilizer subgroup in $SL_2(\mathbb{R})$ of the point $i \in \mathcal{H}$.

Let $g \to g^*$ be conjugate transpose:

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^* = \begin{pmatrix}
\overline{a} & \overline{c} \\
\overline{b} & \overline{d}
\end{pmatrix}
$$

and put $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

[5.0.3] Claim: The subgroup

$$
SU(1,1) = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) : g^*Sg = S, \ det g = 1 \} \subset GL_2(\mathbb{C})
$$

stabilizes the open unit disk and acts transitively on it.

Proof: Observe that

$$
-|\alpha|^2 + |\beta|^2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* S \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$,

$$
(|gz|^2 - 1) \cdot |cz + d|^2 = \left| \begin{pmatrix} az + b \\ cz + d \end{pmatrix} \right|^2 - 1 \cdot |cz + d|^2 = |az + b|^2 - |cz + d|^2
$$

$$
= \begin{pmatrix} az + b \\ cz + d \end{pmatrix}^* S \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \left( g \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^* S \left( g \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = \begin{pmatrix} z \\ 1 \end{pmatrix}^* g^*Sg \begin{pmatrix} z \\ 1 \end{pmatrix}
$$

$$
= \begin{pmatrix} z \\ 1 \end{pmatrix}^* S \begin{pmatrix} z \\ 1 \end{pmatrix} = |z|^2 - 1 < 0 \quad \text{(for $z$ in the unit disk)}
$$

This proves that $U(1,1)$ stabilizes the open disk. There are rotations $z \to \mu^2 \cdot z$ given by $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ in $SU(1,1)$ for $|\mu| = 1$, and these are transitive on each circle of radius $0 \leq r < 1$. The elements

$$
\begin{pmatrix}
\cosh t & \sinh t \\
\sinh t & \cosh t
\end{pmatrix}
$$

are in $SU(1,1)$, and send

$$
0 \to \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2t} - 1}{e^{2t} + 1} \quad \text{(for $t \in \mathbb{R}$)}
$$
By the intermediate value theorem, since \( \lim_{t \to +\infty} \tanh t = 1 \), every real value \( T \) in the interval \( 0 \leq T < 1 \). Thus, \( SU(1,1) \) is transiton on the open unit disk.

[5.0.4] **Remark:** Conjugation by the Cayley map \( f = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \) has the property that \( C \cdot g \cdot C^{-1} \in SL_2(\mathbb{R}) \) for \( g \in SU(1,1) \), and *vice-versa*. In particular, \( C^{-1} \) conjugates rotations of the disk to the group \( SO_2(\mathbb{R}) \).

[5.0.5] **Corollary:** *(of Schwarz’ lemma)* Any holomorphic automorphism of the open unit disk is given by an element of \( SU(1,1) \). Any holomorphic automorphism of the upper half-plane is given by an element of \( SL_2(\mathbb{R}) \).

**Proof:** Given a holomorphic map \( f \) of the open disk to itself, compose with an element of \( U(1,1) \) to adjust so that \( f(0) = 0 \). Certainly \( |f(z)| \leq 1 \) for \( |z| < 1 \). By Schwarz’ lemma, \( |f(z)| \leq |z| \) and \( |f'(0)| \leq 1 \), and if equality holds at any point with \( 0 < |z| < 1 \) or if \( |f'(0)| = 1 \), then \( f(z) = \mu \cdot z \) with \( |\mu| = 1 \).

By the holomorphic inverse function theorem, \( (f^{-1})'(0) = 1/f'(0) \). Also, Schwarz’ lemma applies to \( f^{-1} \). Thus,

\[
1 \leq \frac{1}{|f'(0)|} = |(f^{-1})'(0)| \leq 1
\]

implies equality, and that \( f(z) = \mu \cdot z \) for some \( |\mu| = 1 \). This shows that the linear fractional transformations given by \( SU(1,1) \) are the whole holomorphic automorphisms of the open unit disk.

Similarly, for a holomorphic automorphism \( f \) of \( \hat{\mathbb{D}} \), let \( f(i) = z \), and let \( g \in SL_2(\mathbb{C}) \) map \( g(z) = i \). Then

\[
h = C^{-1} \circ g \circ f \circ C
\]

is a holomorphic automorphism of the open unit disk fixing 0, so is a rotation coming from \( SU(1,1) \). Note that \( CgC^{-1} \in SO_2(\mathbb{R}) \). Then

\[
f = g (C \circ h \circ C^{-1})
\]

expresses \( f : \hat{\mathbb{D}} \to \hat{\mathbb{D}} \) as a composition of linear fractional transformations \( g \in SL_2(\mathbb{R}) \).

[5.0.6] **Remark:** The action on \( \mathbb{CP}^1 \) given by *scalar* elements of \( GL_2(\mathbb{C}) \) is *trivial*, so the central subgroups of \( U(1,1) \) and of \( SL_2(\mathbb{R}) \) act trivially.

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### 6. Schwarz’ lemma

[6.0.1] **Theorem:** For \( f \) holomorphic on \( |z| < 1 \), with bound \( |f(z)| \leq 1 \), and with \( f(0) = 0 \),

\[
|f(z)| \leq |z| \quad \text{and} \quad |f'(0)| \leq 1
\]

Equality \( |f(z_0)| = |z_0| \) holds for some \( 0 < |z_0| < 1 \) if and only if \( f(z) = \mu \cdot z \) for some constant \( \mu \) with \( |\mu| = 1 \).

**Proof:** The function \( F(z) = f(z)/z \) has a removable singularity at \( z = 0 \), and takes value \( f'(0) \) there. For each \( 0 < r < 1 \),

\[
|F(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r} \quad \text{(on the circle } |z| = r \text{)}
\]

By the maximum modulus principle, \( |F(z)| \leq 1/r \) on \( |z| \leq r \). Thus, for all \( r' \) with \( 0 < r < r' < 1 \), \( |F(z)| \leq 1/r' \) on \( |z| \leq r \), so \( |F(z)| \leq 1 \) on \( |z| \leq r \). This holds for every \( 0 < r < 1 \), so \( |F(z)| \leq 1 \) on \( |z| < 1 \). This already gives \( f'(0) \leq 1 \).

If \( |F(z_0)| = 1 \) for some \( 0 < |z_0| < 1 \), or if \( |f'(0)| = 1 \), then \( F \) attains its sup in the interior, so is a constant \( \mu \), and \( f(z) = \mu \cdot z \).

///