(*December 22, 2014*)

Partial fractions and prescribed pole data

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/complex/08b_partial_fractions.pdf]

---

1. *Partial fraction expansion of rational functions*

We recall the mechanisms involved in *partial fraction* expansion of rational functions.

When a rational function is (re-) written as the sum of a polynomial and non-polynomial terms of the simple form $c/(z-z_o)^n$, differentiation and integration are much easier.

Even in this case, concepts from complex analysis supplement the elementary algebra. For example, with distinct a, b, knowing that there *exist* A, B such that

$$\frac{1}{(z-a)(z-b)} \;=\; \frac{A}{z-a} + \frac{B}{z-b}$$

we realize that the residues of the left-hand side at the two poles are exactly the two constants: at $z = a$, the residue is $1/(a-b)$, and at $z = b$ the residue is $1/(b-a)$:

$$\frac{1}{(z-a)(z-b)} \;=\; \frac{1/(a-b)}{z-a} + \frac{1/(b-a)}{z-b}$$

The same device works with larger products of distinct linear factors. With repeated factors, the advantage of this viewpoint is somewhat reduced, but the viewpoint is still helpful.

Indeed, although the notion of *residue* most often appears in the context of elementary complex analysis, it can also be given a comparable sense for algebraic curves and rational functions on them.

From an elementary algebraic viewpoint: for Q, R relatively prime polynomials in the *Euclidean ring* [1] $\mathbb{C}[X]$, there are polynomials A, B such that $A \cdot Q + B \cdot R = 1$. Thus, for example, with another polynomial R,

$$\frac{P(z)}{Q(z)\cdot R(z)} \;=\; \frac{P(z)\cdot\big(A(z)\cdot Q(z) + B(z)\cdot R(z)\big)}{Q(z)\cdot R(z)} \;=\; \frac{P(z)\cdot A(z)}{Q(z)} + \frac{P(z)\cdot B(z)}{R(z)}$$

Since $\mathbb{C}$ is algebraically closed, polynomials can be factored, up to constants, into products of powers $(z-z_o)^n$ of distinct linear factors $z - z_o$. Thus,

$$\text{(rational function)} \;=\; \text{(sum of terms of the form)}\; \frac{P(z)}{(z-z_o)^n} \qquad\qquad \text{(polynomial } P\text{)}$$

By polynomial division, the polynomial P can be expressed as

$$P(X) \;=\; c_0 + c_1(X-z_o) + c_2(X-z_o)^2 + \ldots + c_{N-1}(X-z_o)^{N-1} + c_N(X-z_o)^N$$

---

[1] A *norm* on a commutative ring R with identity 1 is a non-negative-real-valued function $r \to |r|$ with $|ab| = |a|\cdot|b|$, $|a+b| \le |a| + |b|$, $|1| = 1$, and $|0| = 0$. Such a ring is *Euclidean* when, given $x \in R$ and $0 \ne d \in R$, there is $q \in R$ such that $|x - d\cdot q| < |d|$. A typical example is $R = \mathbb{Z}$ with the usual absolute value. This property assures that the Euclidean algorithm succeeds, which implies that the ring is a *principal ideal domain*, and a *unique factorization domain*.

Then

$$\frac{P(z)}{(z-z_o)^n} \;=\; \frac{c_0}{(z-z_o)^n} + \frac{c_1}{(z-z_o)^{n-1}} + \ldots + \frac{c_{N-n+1}}{z-z_o} + \Big(c_{N-n} + c_{N-n-1}(z-z_o) + \ldots + c_N(z-z_o)^{N-n}\Big)$$

Thus, with distinct z_j and positive integers e_j,

$$\frac{\text{polynomial}}{(z-z_1)^{e_1}\cdot(z-z_m)^{e_m}} \;=\; \text{polynomial} + \sum_{1\le i\le m}\;\sum_{1\le j\le e_i} \frac{C_{ij}}{(z-z_i)^j}$$

for constants C_{ij} and some leftover polynomial.

---

2. *Meromorphic functions on $\mathbb{C}$ with prescribed pole data*

Perhaps surprisingly, there is a meromorphic function on $\mathbb{C}$ with prescribed finite Laurent expansions at any discrete set of points in $\mathbb{C}$:

[2.0.1] Theorem: Given a *discrete* set of points $\{z_1, z_2, \ldots\} \subset \mathbb{C}$, and polynomials $P_j(X) \in \mathbb{C}[X]$ for $j = 1, 2, \ldots$ such that $P_j(0) = 0$, there is a function f meromorphic on $\mathbb{C}$ with poles at the z_j, such that

$$f(z) \;=\; P_j\Big(\frac{1}{z-z_j}\Big) + (\text{power series in } z-z_j) \qquad\qquad (\text{Laurent expansion near } z_j)$$

In particular, there are polynomials Q_j such that the expression

$$f(z) \;=\; \sum_j \Big(P_j\Big(\frac{1}{z-z_j}\Big) + Q(z-z_j)\Big)$$

converges absolutely and uniformly on compacts.

[2.0.2] Remark: Often the particulars of specific examples allow a sharper and explicit form of the conclusion.

Proof: The natural first approximation to a meromorphic function on $\mathbb{C}$ with the prescribed polar parts would be

$$\sum_j P_j\Big(\frac{1}{z-z_j}\Big) \qquad\qquad (\text{natural guess to make polar parts } P_j(1/(z-z_j)))$$

Although the points z_j are presumed to have no limit point in $\mathbb{C}$, this natural sum may fail to converge. For example,

$$\sum_{n\in\mathbb{Z}} \frac{n}{z-n} \qquad \text{and} \qquad \sum_{m,n\in\mathbb{Z}} \frac{1}{z-(m+in)}$$

are both divergent. As suggested in the statement of the theorem, it is possible to adjust by polynomials to compensate, as follows.

For simplicity, suppose that 0 is not in the discrete set of required poles. Each $P_j(1/(z-z_j))$ is holomorphic near 0, so has a convergent power series expansion $\sum_{n\ge 0} c_{jn}z^n$ at 0. Indeed, this power series has radius of convergence $|z_j|$, so $|c_{jn}| \le C_j \cdot |1/z_j|^n$ for some constant, by the *root test*. In particular, the *remainder* after the N^{th} term is

$$\Big|P_j\Big(\frac{1}{z-z_j}\Big) - \sum_{0\le n\le N} c_{jn}z^n\Big| \;=\; \Big|\sum_{n>N} c_{jn}z^n\Big| \;\le\; C_j \sum_{n>N} \Big|\frac{z}{z_j}\Big|^n \;\le\; C_j \frac{|z/z_j|^{N+1}}{1-|z/z_j|}$$

Taking $|z|$ less than half the distance to the z_j nearest 0, this is bounded by

$$C_j \sum_{n>N} \Big|\frac{z_j/2}{z_j}\Big|^n \;=\; C_j \cdot \frac{2^{-(N+1)}}{1-\frac{1}{2}} \;=\; C_j \cdot 2^{-N}$$

By taking $N = N_j$ large enough, the sum over j of all these remainders is uniformly absolutely convergent in the disk where $|z|$ is less than half the smallest $|z_j|$, so gives a holomorphic function on that disk. Also, with $\mu = \inf_j |z|_j$,

$$\Big|\sum_j P_j\big(\frac{1}{z-z_j}\big) - \sum_{0\le n\le N_j} c_{jn} z^n\Big| \;\le\; \sum_j C_j \Big|\frac{1}{z_j}\Big|^{N_j} \cdot |z|^{N_j} \qquad\qquad (\text{for all } |z| \le \tfrac{1}{2}\mu)$$

Without loss of generality, $N_1 < N_2 < N_3 < \ldots$. By the *ratio test*, the latter sum converges for all z when

$$\lim_j \Big(C_j \cdot \Big|\frac{1}{z_j}\Big|^{N_j}\Big)^{1/N_j} \;=\; \lim_j C_j^{1/N_j} \cdot \frac{1}{|z_j|} \;=\; 0$$

Since $|z_j| \to +\infty$, there exists a choice of $\{N_j\}$ achieving this effect. Thus, with

$$Q_j(z) \;=\; \sum_{n\le N_j} c_{jn} z^n$$

the series

$$\sum_j \Big|P_j\big(\frac{1}{z-z_j}\big) + Q(z-z_j)\Big|$$

is dominated by a power series convergent on $\mathbb{C}$. However, the series involving the P_j is not a power series, so something further is necessary.

Given $r > 0$, the set $J = \{j : |z_j| \le r\}$ is *finite*, by discreteness of $\{z_j\}$. Then the ratio-test bound implies that the power series for

$$\sum_{j\notin J} \Big(P_j\big(\frac{1}{z-z_j}\big) + Q(z-z_j)\Big)$$

is absolutely and uniformly convergent in $|z| \le r$. Adding the finitely-many terms for $j \in J$ creates no convergence problems, although introducing finitely-many poles in that disk. This applies to every $r < \infty$, so the indicated series converges (uniformly on compacts not meeting the set of poles) and gives a meromorphic function on $\mathbb{C}$. ///

[2.0.3] Remark: Again, simplifications convenient for treatment of the general case, such as invocation of the root test by assuming $N_1 < N_2 < \ldots$, can be avoided in many specific examples.
