(*January 20, 2015*)

Simple Proof of the Prime Number Theorem

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

The assertion of *the Prime Number Theorem* is that

$$\lim_{x\to\infty} \frac{\text{number of primes} \le x}{x/\log x} \;=\; 1$$

This is usually written as

$$\pi(x) \;\sim\; \frac{x}{\log x}$$

A form of this was conjectured by Gauss about 1800, [Chebyshev 1848/52] and [Chebyshev 1850/52] made notable progress with essentially elementary methods. The landmark paper Riemann 1859] made clear the intimate connection between prime numbers and the behavior of $\zeta(s)$ as a function of a complex variable. The theorem was proven independently by [Hadamard 1896] and [de la Vallée Poussin 1896] by complex-analytic methods.

Other proofs in the early 20th century mostly used *Tauberian theorems* , as in [Wiener 1932], to extract the Prime Number Theorem from the non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$.

[Erdos 1950] and [Selberg 1950] gave proofs of the Prime Number Theorem *elementary* in the sense of using no complex analysis or other limiting procedure devices. At the time, it was hoped that this might shed light on the behavior of the zeta function, since the latter had proven more difficult than anticipated in the late 19th century. However, these elementary methods did not give the expected sharp error term.

[Newman 1980] gave a very simple proof that non-vanishing of the zeta function $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$ implies the Prime Number Theorem, avoiding estimates on the zeta function at infinity and avoiding Tauberian arguments.

For completeness, we recall the standard *ad hoc* argument for the non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$, thus giving a complete proof of the Prime Number Theorem. The discussion below is a minor modification of part of [Garrett 2000], which shows how to apply the idea of [Newman 1980] to a larger class of Dirichlet series and corresponding number-theoretic asymptotics.

This gives the clearest proof of the Prime Number Theorem itself, but does not explain the relation between *zero-free regions* and the *error term* in the Prime Number Theorem.

1. Non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$
2. Convergence theorem
3. First corollary on asymptotics
4. Elementary lemma on asymptotics
5. The Prime Number Theorem
6. Second corollary on asymptotics

1. *Non-vanishing of* $\zeta(s)$ *on* $\mathrm{Re}(s) = 1$

It is highly non-trivial to see that the Riemann zeta function

$$\zeta(s) \;=\; \sum_n \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

does not vanish on the line $\mathrm{Re}(s) = 1$. This non-vanishing was not proven until 1896, independently by Hadamard and de la Valleé-Poussin. The by-now standard elementary but *ad hoc* proof is given below, for completeness. As a consequence, the logarithmic derivative of the *Euler product*

$$\frac{d}{ds} \log \zeta(s) \;=\; \frac{\zeta'(s)}{\zeta(s)} \;=\; -\sum_p \frac{d}{ds} \log(1 - p^{-s}) \;=\; -\sum_p \frac{\log p}{p^s} - \sum_p \sum_{m \geq 2} \frac{\log p}{p^{ms}}$$

is *holomorphic* except for the pole at $s = 1$ on an open set containing $\mathrm{Re}(s) \geq 1$. From this we prove (below) the *Prime Number Theorem*

$$\lim_{x \to \infty} \frac{\text{number of primes} \leq x}{x / \log x} \;=\; 1$$

or, at it is usually written,

$$\pi(x) \;\sim\; \frac{x}{\log x}$$

[1.0.1] Proposition: $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 1$.

Proof: For arbitrary real θ

$$3 + 4\cos\theta + \cos 2\theta \;\geq\; 0$$

because $\cos 2\theta = 2\cos^2\theta - 1$ and then

$$3 + 4\cos\theta + \cos 2\theta \;=\; 2 + 4\cos\theta + 2\,\cos^2\theta \;=\; 2\,(1 + \cos\,\theta)^2 \;\geq\; 0$$

Suppose that $\zeta(1 + it) = 0$, and consider

$$D(s) \;=\; \zeta(s)^3 \cdot \zeta(s + it)^4 \cdot \zeta(s + 2it)$$

At $s = 1$ the pole of $\zeta(s)$ at $s = 1$ would cancel some of the alleged vanishing of $\zeta(s + it)$ at $s = 1$, and $1 + 2it$ might be a 0 of $\zeta(s)$, but not a pole. Thus, if $D(s)$ does not have a zero at $s = 1$, then $\zeta(s)$ has no zero on $\mathrm{Re}(s) = 1$.

For $\mathrm{Re}(s) > 1$, taking the logarithmic derivative of $D(s)$ gives

$$\frac{d}{ds} \log D(s) \;=\; -\sum_p \sum_{m \geq 1} \frac{(3 + 4p^{-mit} + p^{-2mit}) \log p}{p^{ms}}$$

The limit of this multiplied by $(s - 1)$, as $s \to 1$ from the right (on the real axis), is the order of vanishing of $D(s)$ at $s = 1$, including as usual poles as negative ordersof vanishing. The real part of $3 + 4p^{-mit} + p^{-2mit}$ is non-negative, as noted. Thus, as $s \to 1$ along the real axis from the right, the real part of the latter expression is non-positive (due to the leading minus sign). In particular, this limit cannot be a positive integer. Thus, $D(s)$ does not have a genuine zero at $s = 1$. As noted, this implies that $\zeta(1 + it) \neq 0$. ///

2. *Convergence theorems*

The first theorem below has more obvious relevance to Dirichlet series, but the second version is what we will use to prove the Prime Number Theorem. A unified proof is given.

[2.0.1] Theorem: *(Version 1)* Suppose that c_n is a bounded sequence of complex numbers. Define

$$D(s) \;=\; \sum_n \frac{c_n}{n^s}$$

Suppose that $D(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \geq 1$. Then the sum $\sum_n \frac{c_n}{n^s}$ converges for $\mathrm{Re}(s) \geq 1$.

[2.0.2] Theorem: *(Version 2)* Suppose that $S(t)$ is a bounded locally integrable complex-valued function. Put

$$f(s) \;=\; \int_0^\infty S(t)\, e^{-st}\, dt$$

Suppose that $f(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \geq 0$. Then the integral $\int_0^\infty S(t)\, e^{-st}\, dt$ converges for $\mathrm{Re}(s) \geq 0$ and equals $f(s)$.

Proof: The boundedness of the constants c_n assures that $D(z) = \sum_n \frac{c_n}{n^z}$ is holomorphic for $\mathrm{Re}(z) > 1$. In this case define $f(z) = D(z+1)$. Thus, in either case we have a function $f(s)$ holomorphic on an open set containing $\mathrm{Re}(z) \geq 0$.

Let $R \geq 1$ be large. Depending on R, choose $0 < \delta < 1/2$ so that $f(z)$ is holomorphic on the region $\mathrm{Re}(z) \geq -\delta$ and $|z| \leq R$, and let $M \geq 0$ be a bound for it on that (compact) region.

Let γ be the (counter-clockwise) path bounded by the arc $|z| = R$ and $\mathrm{Re}(z) \geq -\delta$, and by the straight line $\mathrm{Re}(z) = -\delta$, $|z| \leq R$. Let A be the part of γ in the right half-plane and let B be the part of γ in the left half-plane.

By residues

$$2\pi i f(0) \;=\; \int_\gamma f(z)\, N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

Indeed, the integral of $f(z)$ against the $N^z z/R^2$ term is simply 0 (by Cauchy's theorem), since $f(z)\cdot N^z z/R^2$ is holomorphic on a suitable region. On the other hand, the integral of $f(z)N^z$ against $1/z$ is $2\pi i$ times the value of $f(z)N^z$ at $z = 0$, which is $f(0)$.

The N^{th} partial sum or truncated integral (respectively)

$$S_N(z) \;=\; \sum_{n<N} \frac{c_n}{n^z} \qquad\qquad S_N(z) \;=\; \int_0^N S(t)\, e^{-zt}\, dt$$

of $f(z)$ is an entire function of z, so we can express $S_N(0)$ as an integral over the whole circle of radius R centered at 0, rather than having to use the path along $\mathrm{Re}(z) = -\delta$ as for $f(z)$, namely

$$2\pi i S_N(0) \;=\; \int_{A\cup -A} S_N(z)\, N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

where $-A$ denotes the left half of the circle of radius R. Breaking the integral into A and $-A$ pieces and replacing z by $-z$ in the $-A$ integral gives

$$\int_A S_N(z) N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz \;=\; 2\pi i S_N(0) - \int_A S_N(-z) N^{-z} \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

On the arc A, $f(z)$ is equal to its defining series, which we split into the N^{th} partial sum $S_N(z)$ and the corresponding N^{th} tail $T_N(z)$. Therefore, the N-tail $T_N(0) = f(0) - S_N(0)$ of the series/integral for $f(0)$ has an expression

$$2\pi i(f(0) - S_N(0)) \;=\; \int_A \big(T_N(z)N^z - S_N(-z)N^{-z}\big)\left(\frac{1}{z} + \frac{z}{R^2}\right) dz + \int_B f(z)N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz$$

Elementary estimates will show that this goes to 0 as N becomes large.

We carry out these estimates in some detail. Use $a \ll b$ to mean $a = O(b)$, and let $x = \mathrm{Re}(z)$. We'll need some obvious and elementary inequalities:

$$\begin{cases} \dfrac{1}{z} + \dfrac{z}{R^2} \;=\; \dfrac{2x}{R^2} & \text{on } |z| = R \\ \dfrac{1}{z} + \dfrac{z}{R^2} \;\ll\; \dfrac{2}{\delta} & \text{along } B, \text{ for fixed } R, \text{ for } \delta \text{ sufficiently small} \\ T_N(z) \;\ll\; \displaystyle\int_N^\infty \frac{dn}{n^{x+1}} = \frac{1}{xN^x} & \\ S_N(-z) \;\ll\; \displaystyle\int_0^N n^{x-1}\, dn = N^x\left(\frac{1}{N} + \frac{1}{x}\right) & \end{cases}$$

On the contour A

$$T_N(z) \cdot N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) \;\ll\; \frac{1}{xN^x} \cdot \frac{2x}{R^2} \ll \frac{1}{R^2}$$

and also on A

$$S_N(z) \cdot N^{-z} \left(\frac{1}{z} + \frac{z}{R^2}\right) \;\ll\; N^x \left(\frac{1}{N} + \frac{1}{x}\right) \cdot \frac{2x}{R^2} \;\ll\; \frac{1}{R^2} + \frac{1}{N\,R}$$

with constants independent of N, R, etc. Thus, estimating the integral over A by the sup of the absolute value of the integrand multiplied by the length of the path,

$$\int_A \big(T_N(z)N^z - S_N(-z)N^{-z}\big)\left(\frac{1}{z} + \frac{z}{R^2}\right) dz \;\ll\; \frac{1}{R} + \frac{1}{N}$$

On the path B,

$$\int_B f(z)\, N^z \left(\frac{1}{z} + \frac{z}{R^2}\right) dz \le \int_B M \cdot N^x \cdot \left(\frac{1}{|z|} + \frac{|z|^2}{R^2}\right) |dz| \le M \cdot \int_{-R}^R N^{-\delta} \cdot \frac{2}{\delta}\, dy + 2 \cdot M \cdot \int_{-\delta}^0 N^x \cdot \frac{1}{R}\, dx$$

$$\le \frac{4M}{\delta N^\delta} + \frac{2M}{R \log N}$$

Thus, altogether,

$$f(0) - S_N(0) \;\ll\; \frac{1}{R} + \frac{1}{N} + \frac{RM}{\delta N^\delta} + \frac{M}{R \log N}$$

In this expression, for given positive ε take $R = 1/\varepsilon$, (with corresponding choice of δ, and then of bound M) obtaining

$$f(0) - S_N(0) \;\ll\; \varepsilon \cdot \left(1 + \frac{1}{\varepsilon N} + \frac{M}{\varepsilon\delta N^\delta} + \frac{M}{\log N}\right)$$

for all N. For sufficiently large N the expression inside the parentheses is smaller than (for example) 2, proving that the sum/integral for $f(0)$ converges by proving that the partial sums/integral $S_N(0)$ converge to the value $f(0)$ of the holomorphic function f at 0. ///

3. *Corollary on asymptotics*

This corollary of the convergence theorem is sufficient to prove the Prime Number Theorem.

[3.0.1] Corollary: Let c_n be a sequence of non-negative real numbers, and let

$$D(s) \;=\; \sum_n \frac{c_n \cdot \log n}{n^s}$$

Suppose

$$S(x) \;=\; \sum_{n\le x} c_n \cdot \log n$$

is $O(x)$, and that $(s-1)D(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \ge 1$. That is, except for a possible simple pole at $s = 1$, $D(s)$ is holomorphic on $\mathrm{Re}(s) \ge 1$. Let ρ be the residue of $D(s)$ at $s = 1$. Then

$$\sum_{n\le x} c_n \cdot \log n \;\sim\; \rho\, x$$

Proof: Integrating by parts, writing the sum as a Stieltjes integral,

$$D(s) \;=\; \int_1^\infty t^{-s}\, dS(t) \;=\; s\cdot \int_1^\infty S(t)\, t^{-s-1}\, dt \;=\; s\cdot \int_0^\infty S(e^t)\, e^{-ts}\, dt$$

by replacing t by e^t. For $\mathrm{Re}(s) > 0$, from the definition

$$\int_0^\infty \big(S(e^t)e^{-t} - \rho\big)\; e^{-st}\, dt \;=\; \frac{f(s+1)}{s+1} - \frac{\rho}{s}$$

Since $S(e^t)e^{-t}$ is bounded, and the right-hand side is holomorphic on an open set containing $\mathrm{Re}(s) \ge 0$, the convergence theorem applies, so

$$\int_0^\infty \big(S(e^t)e^{-t} - \rho\big)\; e^{-st}\, dt$$

is convergent for $\mathrm{Re}(s) \ge 0$. In particular, the integral for $s = 0$, namely

$$\int_0^\infty \big(S(e^t)e^{-t} - \rho\big)\; dt$$

is convergent. Changing variables back, replacing e^t by t,

$$\int_1^\infty \frac{S(t) - \rho t}{t^2}\, dt$$

is convergent.

To complete the proof, note that $S(x)$ is positive real-valued and non-decreasing. Suppose there is $\varepsilon > 0$ so that there exist arbitrarily large x with $S(x) > (1+\varepsilon)\rho x$. Then

$$\int_x^{(1+\varepsilon)x} \frac{S(t) - \rho t}{t^2}\, dt \ge \int_x^{(1+\varepsilon)x} \frac{(1+\varepsilon)\rho x - \rho t}{t^2}\, dt \;=\; \rho\cdot \int_1^{1+\varepsilon} \frac{(1+\varepsilon) - t}{t^2}\, dt$$

by replacing t by tx, using the non-decrease of $S(x)$. For $\rho \neq 0$, the latter expression is strictly positive and does not depend upon x, contradicting the convergence of the integral. Similarly, suppose that there is $\varepsilon > 0$ so that there exist arbitrarily large x with $S(x) < (1-\varepsilon)\rho x$. Then

$$\int_{(1-\varepsilon)x}^{x} \frac{S(t)-\rho t}{t^2}\,dt \le \int_{(1-\varepsilon)x}^{x} \frac{(1-\varepsilon)\rho x - \rho t}{t^2}\,dt \;=\; \rho \cdot \int_{1-\varepsilon}^{1} \frac{(1-\varepsilon)-t}{t^2}\,dt$$

which is negative and independent of x, contradicting the convergence. ///

---

4. *Elementary lemma on asymptotics*

The lemma here is elementary but used over and over, so deserves to be understood clearly apart from other issues.

[4.0.1] Lemma: Let $f(x)$ be some function and suppose that

$$\sum_{p\le x} f(p)\cdot \log p \;\sim\; rx$$

Then

$$\sum_{p\le x} f(p) \;\sim\; \frac{rx}{\log x}$$

Proof: Let

$$\theta(x) \;=\; \sum_{p\le x} f(p)\cdot \log p \qquad\qquad \varphi(x) \;=\; \sum_{p\le x} f(p)$$

Use a '*' to denote a sufficiently large but fixed lower limit of integration, whose precise nature is irrelevant to these asymptotic estimates. Integrating by parts,

$$\varphi(x) \sim \int_*^x d\varphi(t) \;=\; \int_*^x \frac{1}{\log t}\cdot d\theta(t) \;=\; \left[\frac{1}{\log t}\theta(t)\right]_*^x + \int_*^x \theta(t)\,\frac{1}{t\log^2 t}\,dt$$

Estimate the integral of $1/\log^2 t$ via

$$\int_*^x \frac{1}{\log^2 t}\,dt \;=\; \int_*^{\sqrt{x}} \frac{1}{\log^2 t}\,dt + \int_{\sqrt{x}}^x \frac{1}{\log^2 t}\,dt$$

$$= \int_*^{\sqrt{x}} \frac{t}{t\log^2 t}\,dt + \int_{\sqrt{x}}^x \frac{1}{\log^2 t}\,dt \le \sqrt{x}\cdot \int_*^{\sqrt{x}} \frac{1}{t\log^2 t}\,dt + \frac{1}{\log^2\sqrt{x}}\cdot \int_{\sqrt{x}}^x 1\,dt$$

$$\sim\; \frac{2\sqrt{x}}{\log x} + \frac{4x}{\log^2 x} \;=\; o\left(\frac{x}{\log x}\right)$$

Thus,

$$\varphi(x) \;\sim\; \frac{rx}{\log x} - \int_*^x \theta(t)\,\frac{1}{t\log^2 t}\,dt \;\sim\; \frac{rx}{\log x}$$

as claimed. ///

5. *The Prime Number Theorem*

This is the simplest example of application of the analytical results above. As always, $\pi(x)$ is the number of primes less than x. Using Chebycheff's notation, let

$$\theta(x) \;=\; \sum_{p<x} \log p$$

[5.0.1] Theorem: *(Prime Number Theorem)*

$$\pi(x) \;\sim\; \frac{x}{\log x}$$

Proof: First, use properties of $\zeta(s)$ and the convergence theorem's corollary to prove that

$$\theta(x) \;\sim\; x$$

Taking the logarithmic derivative of the zeta function gives

$$\frac{d}{ds}\log\zeta(s) = \frac{\zeta'(s)}{\zeta(s)} \;=\; -\sum_p \frac{d}{ds}\log(1-p^{-s}) \;=\; -\sum_p \frac{\log p}{p^s} - \sum_p \sum_{m\geq 2} \frac{\log p}{p^{ms}}$$

The second sum in the latter expression is readily estimated to give a holomorphic function in the region $\mathrm{Re}(s) > \frac{1}{2}$, so the non-vanishing of $\zeta(s)$ on $\mathrm{Re}(s) = 1$ (and the simple pole with residue 1 at $s = 1$) implies that the function

$$f(s) \;=\; \sum_p \frac{\log p}{p^s}$$

has a simple pole with residue 1 at $s = 1$ and is otherwise holomorphic on $\mathrm{Re}(s) \geq 1$. This Dirichlet series has coefficients

$$c_n \;=\; \begin{cases} \log p & \text{(for } n = p \text{ prime)} \\ \\ 0 & \text{(otherwise)} \end{cases}$$

The corollary of the convergence theorem gives $\sum_{p\leq x} \log p \sim x$, and application of the lemma above gives the asserted asymptotic for $\pi(x)$. ///

6. *Second corollary on asymptotics*

[6.0.1] Corollary: Let c_p be a *bounded* sequence of complex numbers indexed by primes p, and let

$$D(s) \;=\; \sum_p \frac{c_p \cdot \log p}{p^s}$$

Let

$$S(x) \;=\; \sum_{p\leq x} c_p \cdot \log p$$

and suppose that $S(x) = O(x)$, and that $(s-1)D(s)$ extends to a holomorphic function on an open set containing the closed set $\mathrm{Re}(s) \geq 1$. That is, except for a possible simple pole at $s = 1$, $D(s)$ is holomorphic on $\mathrm{Re}(s) \geq 1$. Let ρ be the residue of $D(s)$ at $s = 1$. Then

$$\sum_{p\leq x} c_p \log p \;\sim\; \rho\, x$$

Proof: First, consider the case that

$$S(x) \;=\; \sum_{p<x} c_p \cdot \log p$$

is real-valued (but not necessarily non-decreasing). Let C be a sufficiently large positive constant so that $C + c_p \geq 0$ for every prime index p. Then the *first* corollary applies to $S_1(x) \;=\; \sum_{p\leq x} (C + c_p) \cdot \log p$ and to the associated Dirichlet series

$$D_1(s) \;=\; \sum_p \frac{(C + c_p)\cdot \log p}{p^s} \;=\; C \cdot \sum_p \frac{\log p}{p^s} + D(s)$$

We already know that $(s-1)\sum_p \frac{\log p}{p^s}$ is holomorphic on $\mathrm{Re}(s) \geq 1$, has a simple pole with residue 1 at $s = 1$. And we have already proven the asymptotic assertion $\sum_{p\leq x} \log p \sim x$ from the first corollary. Thus,

$$\sum_{p\leq x} (C + c_p)\cdot \log p \;\sim\; (C + \rho)\cdot x$$

from which

$$\sum_{p\leq x} c_p \cdot \log p \;\sim\; \rho \cdot x$$

by subtracting the asymptotics for $\sum_p \frac{\log p}{p^s}$. This proves the corollary for real-valued bounded c_p. For complex-valued bounded c_p, break everything into real and imaginary parts. ///

[Chebyshev 1848/52] P.L. Chebyshef, *Sur la function qui détermine la totalité des nombres premiers inferieurs à une limite donnée*, J. Math. Pures Appl. **17** (1852) (also published 1848).

[Chebyshev 1850/52] P.L. Chebyshef, *Mémoire sur les nombres premiers*, J. Math. Pures Appl. **17** (1852) (otherwise published 1850).

[Erdos 1949] P. Erdos, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, Proc. Nat. Acad. Sci. USA **35** (1949), 374-384.

[Garrett 2000] P. Garrett, *A simple proof of the prime number theorem, etc.*, (invited paper) Bombay Math. Coll. **16**, no. 1, 2000.

[Hadamard 1896] J. Hadamard, *Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques*, Bull. Soc. Math. France **24** (1896), 199-220.

[Newman 1980] D.J. Newman, *Simple Analytic Proof of the Prime Number Theorem*, Amer. Math. Monthly 87 (1980), no. 7, pp. 693-96.

[de la Vallée-Poussin 1896] C.-L. de la Vallée-Poussin, *Recherches analytiques sur la théorie des nombres premiers*, Annales de la Société Scientifiques de Bruxelles 20B (1896), 183-256.

[Riemann 1859] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monats. Akad. Berlin (1859), 671-680.

[Selberg 1949] A. Selberg, *An elementary proof of the prime number theorem*, Ann. of Math. **50** (1949), 305-313.

[Wiener 1932] N. Wiener, *Tauberian theorems*, Ann. of Math. **33** (1932), 1-100.
