Riemann’s and $\zeta(s)$

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1. Riemann’s explicit formula
2. Analytic continuation and functional equation of $\zeta(s)$
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[Riemann 1859] exhibited a precise relationship between primes and zeros of $\zeta(s)$. A similar idea applies to any zeta or L-function with analytic continuation, functional equation, and Euler product.

It took more than 40 years for [Hadamard 1893], [von Mangoldt 1895], and others to complete Riemann’s sketch of the Explicit Formula relating primes to zeros of the Euler-Riemann zeta function. The idea is that equating the Euler product and Riemann-Hadamard product for zeta allows extraction of an exact formula for a weighted counting of primes in terms of a sum over zeros of zeta. [1]

An essential supporting point is meromorphic continuation of $\zeta(s)$ via integral representation(s) of $\zeta(s)$ in terms of theta function(s). [2] Further, these integral representations give vertical growth estimates, allowing invocation of Hadamard’s theorem on product expansions of entire functions.

A key in analytic continuation and functional equation of $\zeta(s)$ is the functional equation of theta series, from the Poisson summation formula, from the representability of smooth functions by their Fourier series.

Asymptotics of $\Gamma(s)$ and the functional equation of $\zeta(s)$ bound the vertical growth of $\zeta(s)$, allowing application of the Hadamard product result.

1. Riemann’s explicit formula

The dramatic [Riemann 1859] on the relation between primes and zeros of the zeta function anticipated many ideas undeveloped in Riemann’s time. Thus, the following sketch, very roughly following Riemann, is not a proof, but exhibits what is needed to produce a proof.

Riemann knew from Euler that $\zeta(s)$ has an Euler product expansion

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}} \quad \text{(for Re } s > 1)$$

As below, [Riemann 1859] proved that $\zeta(s)$ has a meromorphic continuation so that $(s - 1)\zeta(s)$ is entire, with $0 = \zeta(0) = \zeta(-2) = \zeta(-4) = \ldots \ [3]$ The negative even integer are the trivial zeros of $\zeta(s)$. Riemann


[2] Theta functions are examples of automorphic forms. For practical purposes, modular form and automorphic form are synonyms, despite some sources’ attempts to insist upon delicately precise meanings.

[3] The vanishing at negative even integers is not clear at all, but will follow from the functional equation. Even so, Euler had already done computations with divergent series that could be interpreted as suggesting this!
imagined that \( \zeta(s) \) has a product expansion in terms of its zeros\(^{[4]} \)

\[
(s - 1) \zeta(s) = e^{a+bs} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{2n} \right) e^{-s/2n} \quad (\rho \text{ non-trivial zero of } \zeta, \text{ for all } s \in \mathbb{C})
\]

[Hadamard 1893] proved this. Then, taking logarithmic derivatives of

\[
(s - 1) \prod_{\rho} \frac{1}{1 - \frac{s}{\rho}} = e^{a+bs} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{2n} \right) e^{-s/2n} \quad (\text{Res} > 1)
\]

using \(-\log(1-x) = x + x^2/2 + x^3/3 + \ldots\) on the left-hand side gives

\[
\frac{1}{s-1} - \sum_{m \geq 1, p} \frac{\log p}{p^ms} = b + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) - \sum_{n} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right)
\]

A slight rearrangement:

\[
\sum_{m \geq 1, p} \frac{\log p}{p^ms} = \frac{1}{s-1} - b - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) - \sum_{n} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right) \quad (\text{for Res} > 1)
\]

Diverging slightly from Riemann’s original treatment, apply the Perron identity\(^{[5]} \) (see Appendix)

\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{Y^s}{s} ds = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{Y^s}{s} ds = \begin{cases} 
1 & \text{(for } Y > 1) \\
0 & \text{(for } 0 < Y < 1)
\end{cases}
\]

\[(\text{for } \sigma > 0)\]

to the log-derivative identity multiplied by \( X^s/s \). Assuming legitimacy of application of the Perron identity term-wise to \( X^s/s \) times the left-hand side,

\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s} \sum_{m,p} \frac{\log p}{p^ms} ds = \sum_{m,p} \log p \cdot \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s} p^{-ms} ds = \sum_{p^n < X} \log p
\]

Assuming legitimacy of using residues term-wise to evaluate \( X^s/s \) times the right-hand side, with \( \sigma > 1 \),

\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s} \left( \frac{1}{s-1} - b - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) - \sum_{n} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right) \right) ds
\]

\[= (X-1) - b - \sum_{\rho} \left( \frac{X^\rho}{\rho} + \frac{1}{\rho} + \frac{1}{\rho} \right) - \sum_{n} \left( \frac{X^{-2n}}{-2n} + \frac{1}{2n} - \frac{1}{2n} \right) = X - (b+1) - \sum_{\rho} \frac{X^\rho}{\rho} + \sum_{n \geq 1} \frac{X^{-2n}}{2n}
\]

This gives [vonMangoldt 1893]’s reformulation of Riemann’s Explicit Formula:

\[
\sum_{p^n < X} \log p = X - (b+1) - \sum_{\rho} \frac{X^\rho}{\rho} + \sum_{n \geq 1} \frac{X^{-2n}}{2n}
\]

\([4]\) Euler’s evaluation of \( \sum_{n} \frac{1}{n^s} \) by imagining (and later proving) \( \sin \pi z = \pi z \prod_{n} (1 - z^2/n^2) \) was well known, as was Euler’s product expansion of the inverse of the Gamma-function \( \Gamma(s) = \int_{0}^{\infty} t^{s-1} e^{-t} \frac{dt}{t} \) as \( \frac{1}{\Gamma(s)} = ze^{\gamma z} \Pi_{n}(1 + \frac{z}{n}) e^{-z/n} \).

\([5]\) Perron’s identity is completely standard by now, but was not part of Riemann’s approach. Invocation of the Perron identity allows a somewhat simpler approach than Riemann’s original, due to von Mangoldt and others.
More precisely, because of the way the Perron integral transform is applied, and the fragility of the convergence,

$$\sum_{p^n < X} \log p = X - (b+1) - \lim_{T \to \infty} \sum_{|\text{Im}(\rho)|<T} \frac{X^\rho}{\rho} + \sum_{n \geq 1} \frac{X^{-2n}}{2n}$$

[1.0.1] **Remark:** As in Riemann's original, the above sketch has gaps. The existence and convergence of the Hadamard product needs **generalities** about Weierstrass-Hadamard product expressions for entire functions of prescribed growth, and **specifics** about the vertical growth of the **analytic continuation** of $\zeta(s)$. The analytic continuation of $\zeta(s)$ is discussed in the next section, and growth properties later. The growth properties depend on Stirling-Laplace asymptotics of the Gamma function $\Gamma(s)$, and the **Phragmén-Lindelöf** theorem [Phragmén-Lindelöf 1908].

[1.1] **Non-trivial zeros $\rho$ of $\zeta(s)$** The convergent Euler product shows that $\zeta(s) \neq 0$ in the half-plane $\text{Re}(s) > 1$. The analytic continuation and functional equation (below), and relatively elementary properties of $\Gamma(s)$ show that the only possible non-trivial zeros are in the **critical strip** $0 \leq \text{Re}(s) \leq 1$. In 1896, Hadamard and de la Vallée-Poussin independently proved that there are no zeros on the edges $\text{Re}(s) = 0, 1$ of the critical strip, and used this to prove the **Prime Number Theorem**. The functional equation shows that if $\rho$ is a non-trivial zero, then $1 - \rho$ is a non-trivial zero. The property $\zeta(s) = \overline{\zeta(s)}$ shows that if $\rho$ is a non-trivial zero, then $\bar{\rho}$ is a non-trivial zero.

[1.2] **The Riemann Hypothesis**

After the main term $X$ in the right-hand side of the explicit formula, the next-largest terms would be the $X^\rho/\rho$ summmands, with $0 \leq \text{Re}(\rho) \leq 1$ due to the Euler product and functional equation. The **Riemann Hypothesis** is that all the non-trivial zeros $\rho$ have $\text{Re}(\rho) = \frac{1}{2}$. With a bound like $T \log T$ on the number of zeros below height $T$, proven later, the Riemann hypothesis is equivalent to an error term of order $X^{\frac{1}{2}+\varepsilon}$ in the Prime Number Theorem, for all $\varepsilon > 0$.

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**2. Analytic continuation and functional equation of $\zeta(s)$**

The following ideas gained publicity and significance from Riemann, but were apparently known earlier, to some degree.

The key is that the completed zeta function has an **integral representation** in terms of an **automorphic form**, the simplest **theta function**. Both the **analytic continuation** and the **functional equation** of zeta follow from this integral representation using a functional equation of the theta function, from **Poisson summation**, from **Fourier series**.

[2.1] **Elementary-but-insufficiently-enlightening argument for analytic continuation** Simple calculus can extend the domain of $\zeta(s)$ as far to the left as we want. The idea is to pay attention to **quantitative** aspects of the integral test. First, by comparison to $\int_1^\infty \frac{dx}{x^s}$, the sum $\zeta(s) = \sum_1^\infty 1/n^s$ converges for $\text{Re}(s) > 1$.

To push this further, it is standard to proceed as follows.

$$\zeta(s) - \frac{1}{s-1} = \zeta(s) - \int_1^\infty \frac{dx}{x^s} = \sum_n \left( \frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s} \right) = \sum_n \left( \frac{1}{n^s} - \frac{1}{s-1} \left[ \frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} \right] \right)$$

Even for complex $s$, we have a Taylor-Maclaurin expansion with **error term**

$$(n+1)^{-s} = \left( n \cdot (1 + \frac{1}{n}) \right)^{-s} = n^{-s} \cdot \left( 1 + \frac{1-s}{n} + O(\frac{1}{n^2}) \right) = \frac{1}{n^{s-1}} - \frac{s-1}{n^s} + O(\frac{s-1}{n^{s+1}})$$
The constant in the big-O term is uniform in \( n \) for fixed \( s \). Thus,

\[
\frac{1}{n^s} - \frac{1}{s-1} \left[ \frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} \right] = \frac{1}{n^s} - \frac{1}{n^{s-1}} + \frac{1}{s-1} O\left( \frac{1}{n^{s+1}} \right) = O\left( \frac{1}{n^{s+1}} \right)
\]

That is, for fixed\(^6\) \( \text{Re}(s) > 0 \), we have absolute convergence of

\[
\sum_n \left( \frac{1}{n^s} - \frac{1}{s-1} \left[ \frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} \right] \right)
\]

in the larger region \( \text{Re}(s) > 0 \).

**[2.1.1] Remark:** Iterating the idea of approximating sums by integrals gives a comparable extension to \( \text{Re}(s) > -\ell \) for all \( \ell \), as Euler already effectively found, systematically by Euler-Maclaurin summation. However, such continuations give no clues about functional equations, and certainly not about Riemann’s explicit formula.

**[2.2] Slight modernization of Riemann’s argument** We update Riemann’s idea to avoid needless artifacts. Both the original and this update are archetypes.\(^7\) Let \( f(x) \) be any very well-behaved function on \( \mathbb{R} \), that is, infinitely differentiable, and it and all its derivatives are rapidly decreasing at infinity. These are Schwartz functions, after [Schwartz 1950/51]. Further, take \( f \) even, that is \( f(-x) = f(x) \). The even Schwartz function \( f \) is a dummy, insofar as only its general properties are used. In effect, Riemann’s choice was the Gaussian \( f(x) = e^{-\pi x^2} \), based on connections to Jacobi’s theta functions, as we see along the way. A theta function\(^8\) associated to the even Schwartz function \( f \) is

\[
\theta_f(y) = \sum_{n \in \mathbb{Z}} f(y \cdot n) \quad \text{(for } y > 0 \text{)}
\]

and associated Gamma function\(^9\)

\[
\Gamma_f(s) = \int_0^\infty t^s f(t) \frac{dt}{t}
\]

\(^6\) In fact, the big-O constant is also uniform for \( s \) in compacts inside \( \text{Re}(s) > 0 \). Thus, the series converges locally uniformly on compacts, so does give a holomorphic function.

\(^7\) Riemann’s original line of argument was brought to completion by [Hecke 1918/20]. Substantial modernization occurred in [Matchett 1946], [Iwasawa 1950/52], [Iwasawa 1952], and [Tate 1950/1967]. In particular, these sources observed that certain details involving theta functions were less essential than previously believed. Nevertheless, the automorphic nature of theta functions was also important in its own right.

\(^8\) Again, Riemann used \( f(u) = e^{-\pi u^2} \), and, consistent with an existing convention at the time, in effect defined

\[
\theta(uy) = \sum_{n \in \mathbb{Z}} f(\sqrt{y} \cdot n) \quad \text{(with Gaussian } f(u) = e^{-\pi u^2})
\]

That is, the argument of \( \theta \) is \( iy \) rather than \( y \), and \( \sqrt{y} \) enters on the right side, rather than \( y \). Further, the Gaussian extends to an entire function, and this theta function extends to a holomorphic function, the simplest Jacobi theta function, on the upper half-plane \( \mathfrak{H} \):

\[
\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 z} \quad \text{(with } z \in \mathfrak{H})
\]

\(^9\) With Gaussian \( f(x) = e^{-\pi x^2} \), this construction gives an exponential multiple of the standard Gamma function at \( \frac{s}{2} \):

\[
\Gamma_f(s) = \int_0^\infty t^s e^{-\pi x^2} \frac{dx}{x} = \frac{1}{2} \int_0^\infty t^{\frac{s}{2}} e^{-\frac{\pi x^2}{2}} \frac{dx}{x} = \frac{1}{2} \pi^{-\frac{s}{2}} \int_0^\infty t^{\frac{s}{2}} e^{-x} \frac{dx}{x} = \frac{1}{2} \pi^{-\frac{s}{2}} \Gamma\left( \frac{s}{2} \right)
\]
First, we have the integral representation, from which will follow the meromorphic continuation and functional equation:

[2.2.1] Proposition: \[ \int_0^\infty y^s \frac{\theta_f(y) - f(0)}{2} \frac{dy}{y} = \Gamma_f(s) \cdot \zeta(s) \quad \text{(for Re}(s) > 1) \]

**Proof:** The \( n = 0 \) (constant) term \( f(0) \) of \( \theta_f(y) \) is the only summand not rapidly decreasing. The even-ness of \( f \) makes the \( \pm n \) terms have equal contributions to \( \theta_f(y) \). Thus, interchanging sum and integral, and replacing \( y \) by \( y/n \),
\[
\int_0^\infty y^s \frac{\theta_f(y) - f(0)}{2} \frac{dy}{y} = \sum_{n \geq 1} \int_0^\infty y^s f(yn) \frac{dy}{y} = \sum_{n \geq 1} n^{-s} \int_0^\infty y^s f(y) \frac{dy}{y} = \sum_{n \geq 1} n^{-s} \Gamma_f(s)
\]
as claimed. ///

[2.2.2] Remark: The measure \( \frac{dy}{y} \) is the natural multiplication-invariant measure on the positive reals.

[2.2.3] Theorem: The completed zeta function \( \Gamma_f(s) \cdot \zeta(s) \) has a meromorphic continuation to \( s \in \mathbb{C} \), and \( s(s - 1) \cdot \Gamma_f(s) \cdot \zeta(s) \) is entire.

[2.2.4] Remark: Repeated integration by parts shows that \( \Gamma_f(s) \) itself has a meromorphic continuation:
\[
\Gamma_f(s) = \int_0^\infty t^s f(t) \frac{dt}{t} = \int_0^\infty \frac{t^{s+1}}{s} f'(t) \frac{dt}{t} = \int_0^\infty \frac{t^{s+2}}{s(s+1)} f''(t) \frac{dt}{t} = \int_0^\infty \frac{t^{s+3}}{s(s+1)(s+2)} f'''(t) \frac{dt}{t} = \ldots
\]
Since all the derivatives of \( f \) are of rapid decay, these expressions give an extension of \( \Gamma_f(s) \) to \( s \in \mathbb{C} \) except for at worst \( s = 0, -1, -2, -3, \ldots \).

**Proof:** Break the integral of the integral representation into two parts:
\[
\Gamma_f(s) \cdot \zeta(s) = \int_1^\infty y^s \frac{\theta_f(y) - f(0)}{2} \frac{dy}{y} + \int_0^1 y^s \frac{\theta_f(y) - f(0)}{2} \frac{dy}{y}
\]
It is not hard to check that \( \frac{\theta_f(y)}{2} - \frac{\theta_f(0)}{2} \) is rapidly decreasing at \( +\infty \), so the integral on \( [1, \infty) \) is absolutely convergent (and uniformly for \( s \) in compacts) for all \( s \in \mathbb{C} \).

The behavior of \( \theta_f(y) \) as \( y \to 0^+ \) is harder to analyze, and is best done by the following device.

The trick is to convert the integral on \([0, 1]\) to an integral over \([1, \infty)\), up to two elementary terms. The new integral over \([1, \infty)\) will involve the theta function \( \theta_f \) attached to the Fourier transform
\[
\hat{f}(\xi) = \int_\mathbb{R} e^{-2\pi i \xi x} f(x) \, dx
\]
of \( f \). We grant for the moment that Fourier transform maps the Schwartz space to itself, as is directly verifiable in concrete examples such as the Gaussian \( f(x) = e^{-\pi x^2} \). Simply by changing variables in the integral, we recall a homogeneity property of the Fourier transform:
\[
\hat{f}(x/y) = \int_\mathbb{R} e^{-2\pi i \xi x/y} f(\xi) \, d\xi = |y| \int_\mathbb{R} e^{-2\pi i \xi} f(y\xi) \, d\xi = |y| \cdot (f \circ y)^{-1}(x)
\]
by replacing \( \xi \) by \( \xi y \) in the integral, where \( (f \circ y)(\xi) = f(y\xi) \). We grant ourselves the standard Poisson summation formula
\[
\sum_{n \in \mathbb{Z}} F(n) = \sum_{n \in \mathbb{Z}} \hat{F}(n) \quad \text{(for Schwartz functions } F)\]
(See the Supplement for proof.) Letting $F(x) = f(yx)$ and using the homogeneity property of Fourier transform, this is

$$\sum_{n \in \mathbb{Z}} f(y \cdot n) = \sum_{n \in \mathbb{Z}} \frac{1}{y} \hat{f} \left( \frac{1}{y} \cdot n \right) \quad \text{for } y > 0$$

Thus,

$$\theta_f(y) = \sum_{n \in \mathbb{Z}} f(yn) = \sum_{n \in \mathbb{Z}} \frac{1}{y} \hat{f}(n) = \frac{1}{y} \cdot \theta_{\hat{f}} \left( \frac{1}{y} \right)$$

This gives a way to flip the interval $[0, 1]$ to $[1, \infty)$, by replacing $y$ by $1/y$, accommodating the anomalous terms for $n = 0$ separately:

$$\int_{0}^{1} y^{s} \frac{\theta_f(y) - f(0)}{2} \frac{dy}{y} = \int_{0}^{1} y^{s} \frac{y^{1/2} \theta_f(1/2) - f(0)}{2} \frac{dy}{y} = \int_{0}^{1} y^{s} \frac{y^{1/2} \theta_f(1/2)}{2} - \frac{1}{y} f(0) \frac{dy}{y}$$

$$= \int_{1}^{\infty} y^{-s} \frac{y \theta_f(y) - y \hat{f}(0)}{2} \frac{dy}{y} + \int_{0}^{1} y^{s} \frac{\hat{f}(0)}{2} \frac{dy}{y}$$

$$= \int_{1}^{\infty} y^{1-s} \frac{\theta_f(y) - \hat{f}(0)}{2} \frac{dy}{y} + \frac{f(0)}{2} \int_{0}^{1} y^{s-1} \frac{dy}{y} - \frac{f(0)}{2} \int_{0}^{1} y^{s} \frac{dy}{y}$$

The integral on $[1, \infty)$ is entire in $s$, since $\theta_{\hat{f}}(y) - \hat{f}(0)$ is rapidly decreasing at $\infty$. The two elementary terms have obvious meromorphic continuations. Thus,

$$\Gamma_f(s) \cdot \zeta(s) = \int_{1}^{\infty} \left( y^{s} \frac{\theta_f(y) - \hat{f}(0)}{2} + y^{1-s} \frac{\theta_f(y) - \hat{f}(0)}{2} \right) \frac{dy}{y} + \frac{\hat{f}(0)}{2} \frac{1}{s-1} - \frac{f(0)}{2} \frac{1}{s}$$

Again, the integral is entire, and the elementary terms give the only poles, which are at $s = 0, 1$. ///

[2.2.5] Remark: The expression

$$\Gamma_f(s) \cdot \zeta(s) = \int_{1}^{\infty} \left( y^{s} \frac{\theta_f(y) - \hat{f}(0)}{2} + y^{1-s} \frac{\theta_f(y) - \hat{f}(0)}{2} \right) \frac{dy}{y} + \frac{\hat{f}(0)}{2} \frac{1}{s-1} - \frac{f(0)}{2} \frac{1}{s}$$

gives a bit more information than the bare statement of the theorem, namely, it tells the residues of the poles at $s = 0, 1$, and shows a certain potential symmetry, as in the following.

For $f$ with $\hat{f} = f$ Riemann’s original symmetrical result is recovered:

[2.2.6] Theorem: (Riemann) The completed zeta function

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s)$$

has an analytic continuation to $s \in \mathbb{C}$, except for simple poles at $s = 0, 1$, and has the functional equation

$$\xi(1-s) = \xi(s)$$

Proof: Various means show that $f(x) = e^{-\pi x^2}$ is its own Fourier transform. Thus, the expression in the proof of the previous theorem becomes symmetrical in $s \leftrightarrow 1-s$, and the artifact of the coefficient of $\frac{1}{2}$ on both sides can be discarded. ///
**[2.2.7] Remark:** The leading factor $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ should not be construed as objectionable in any way, but, rather, as something that really does belong with $\zeta(s)$. The $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ is called the **gamma factor** for $\zeta(s)$. In the context of the *Euler product* the modern viewpoint is that the gamma factor is a further Euler factor corresponding to the *prime* $\infty$.

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### 3. Appendix: Perron identity

These contour-integral identities extract information from spectral identities and function-theoretic identities. One spectral identity is transformed into another, by a Fourier transform. Choices are made to heighten an *asymmetry*, wherein one side is seemingly elementary, and the other is whatever it must be.

**[3.1] Heuristic** The best-known identity starts from the idea that for $\sigma > 0$

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{X^s}{s} \, ds = \begin{cases} 1 & \text{(for } X > 1) \\ 0 & \text{(for } 0 < X < 1) \end{cases}$$

(convergence?)

The idea of the proof of this identity is that, for $X > 1$, the contour of integration slides indefinitely to the left, eventually vanishing, picking up the residue at $s = 0$, while for $0 < X < 1$, the contour slides indefinitely to the right, eventually vanishing, picking up no residues.

The idea of the application is that this identity can extract *counting* information from a meromorphic continuation of a Dirichlet series: for example, from

$$\sum_{n} \frac{a_n}{n^s} = f(s) \quad \text{(left-hand side convergent for Re}s > 1)$$

we would have

$$\sum_{n \leq X} a_n = \text{sum of residues of } X^s f(s)/s$$

That is, the *counting* function $\sum_{n \leq X} a_n$ is extracted from the analytic object $\sum_{\lambda} a_{n}/n^s$ by the contour integration. With $f$ a logarithmic derivative, such as $f(s) = \zeta'(s)/\zeta(s)$, the poles of $f$ are mostly the zeros of $\zeta$.

However, the tails of these integrals are fragile.

**[3.2] Simple precise assertion** The elegant simplicity of the idea about moving lines of integration must be elaborated for correctness: for fixed $\sigma > 0$, for $T > 0$, we claim that

$$\int_{\sigma-iT}^{\sigma+iT} \frac{X^s}{s} \, ds = \begin{cases} 1 + O_s\left(\frac{X^\sigma}{T \log X}\right) & \text{(for } X > 1) \\ O_s\left(\frac{X^\sigma}{T \log X}\right) & \text{(for } 0 < X < 1) \end{cases}$$

The proof is a precise form of the idea of sliding vertical contours. That is, for $X > 1$, consider the contour integral around the rectangle with right edge $\sigma \pm iT$, namely, with vertices $\sigma - iT, \sigma + iT, -B + iT, -B - iT$, with $B \to +\infty$. For $0 < X < 1$ consider the contour integral around the rectangle with left edge $\sigma \pm iT$, namely, with vertices $\sigma - iT, \sigma + iT, B + iT, B - iT$, with $B \to +\infty$.

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[10] An insight of modern times is that the completion $\mathbb{R}$ should whenever possible be put on an even footing with the other $p$-adic completions $\mathbb{Q}_p$ of $\mathbb{Q}$. Thus, although there is no actual prime $\infty$ in $\mathbb{Z}$ (or anywhere else), the objects that accompany genuine primes $p$ and completions $\mathbb{Q}_p$ often have analogues for $\mathbb{R}$, so we *backform* to refer to the *prime* $\infty$. One attempt to be less bold in this regard is to speak of *places* rather than *primes*, but there’s little point in fretting about this.
For both $X > 1$ and $0 < X < 1$, the $\pm(B \pm iT)$ edge of the rectangle is dominated by
\[
\int_{-T}^{T} e^{-u|\log X|} \frac{du}{|B \pm iT|} \ll T \cdot \frac{e^{-u|\log X|}}{B} \to 0 \quad \text{as } B \to +\infty
\]
in both cases, the top and bottom edges of the rectangle are dominated by
\[
X^\sigma \cdot \int_{0}^{\infty} e^{-u|\log X|} \frac{du}{((\sigma \pm u) + iT)} \ll X^\sigma \cdot \int_{0}^{\infty} e^{-u|\log X|} \frac{du}{T} \ll \frac{X^\sigma}{T} \cdot |\log X|
\]
This proves the claim. Replacing $X$ by $e^X$ in the estimate gives the equivalent
\[
\frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} e^{sX} s \ ds = \begin{cases} 
1 + O_{T}(\frac{e^{X}}{TX}) & \text{for } X > 0 \\
O_{T}(\frac{e^{X}}{TX}) & \text{for } X < 0
\end{cases}
\]

[3.3] Hazards When the quantity $X$ above is summed, especially if the summation is over a set whose precise specifications are difficult, the denominators of the big-O error terms may blow up. In situations such as
\[
\frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} (\sum_{j} a_{j} e^{-sX_{j}}) e^{sX} \frac{ds}{s} = \sum_{j : X_{j} < X} a_{j} + \sum_{j} a_{j} \cdot O_{T}(\frac{e^{(X - X_{j})}}{TX - X_{j}})
\]
the distribution of the values $X_{j}$ has an obvious effect on the convergence of the error term.

[3.4] The other side of the equation A desired and plausible conclusion such as
\[
\lim_{T} \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} f(s) e^{sX} s \ ds = (\text{sum of } \text{Res}_{s=\rho} f(s) \cdot \frac{e^{X}}{\rho})
\]
summed over poles $\rho$ of $f$ in the left half-plane $\text{Re} < \sigma$, requires that the contour integrals over the other three sides of the rectangle with side $\sigma \pm iT$ go to 0, and that the tails of the vertical integral go to 0. The integral over the large rectangle will be evaluated with $X$ large positive, so the decay condition applies to $f$ to the left. The left side of the rectangle will go to 0 for large enough positive $X$ when $f(s)$ has at worst exponential growth to the left, that is, when $f(s) \ll e^{-C|\text{Re} s|}$ for some large-enough $C$ and $\text{Re} s \to -\infty$. The top and bottom are more fragile, since $e^{X}/s$ does not have strong decay vertically.

Not unexpectedly, the poles of $f$ near $\sigma + iT$ may bunch up as $T$ grows, so that a contour integral must be threaded between them, and the corresponding integral will be somewhat larger simply because of proximity to these poles. This contribution to vertical growth of $f$ is significant in examples.

[3.5] Variant identities When $X/s$ is altered to help convergence of the integral against the counting aspect is inevitably altered. The proofs of variants follow the same straightforward line as above for the simplest case. For $\theta > 0$ and $1 \leq \ell \in \mathbb{Z}$,

\[
\frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \frac{X^{s}}{s((s + \theta)(s + 2\theta)\ldots(s + \ell\theta))} ds = \begin{cases} 
\frac{1}{\ell!} (1 - X^{-\theta})^{\ell} + O_{T}(\frac{X^{\sigma}}{T^{\sigma}|\log X|}) & \text{for } X > 1 \\
O_{T}(\frac{X^{\sigma}}{T^{\sigma}|\log X|}) & \text{for } 0 < X < 1
\end{cases}
\]
Indeed, the residues at the poles $0, -\theta, -2\theta, \ldots, -\ell\theta$ sum to

\[
\begin{align*}
\frac{X^{0}}{(0 + \theta)(0 + 2\theta)\ldots(0 + (\ell - 1)\theta)(0 + \ell\theta)} & + \frac{X^{-\theta}}{(-\theta + 0)(-\theta + 2\theta)\ldots(-\theta + (\ell - 1)\theta)(-\theta + \ell\theta)} \\
& + \frac{X^{-2\theta}}{(-2\theta + 0)(-2\theta + 2\theta)\ldots(-2\theta + (\ell - 1)\theta)(-2\theta + \ell\theta)} \ldots + \frac{X^{-\ell\theta}}{(-\ell\theta + 0)(-\ell\theta + 2\theta)\ldots(-\ell\theta + (\ell - 1)\theta)(-\ell\theta + \ell\theta)} \\
& = \frac{1}{\ell! \theta^{\ell}} - \frac{X^{-\theta}}{1!(\ell - 1)! \theta^{\ell}} + \frac{X^{-2\theta}}{2!(\ell - 2)! \theta^{\ell}} \ldots \pm \frac{X^{-\ell\theta}}{\ell! \theta^{\ell}} = \frac{(1 - X^{-\theta})^{\ell}}{\ell! \theta^{\ell}}
\end{align*}
\]
Bibliography


