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Riemann's and $\zeta(s)$

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[Riemann 1859] exhibited a *precise* relationship between primes and zeros of $\zeta(s)$. A similar idea applies to *any* zeta or L-function with *analytic continuation*, *functional equation*, and *Euler product*.

It took more than 40 years for [Hadamard 1893], [vonMangoldt 1895], and others to complete Riemann's sketch of the *Explicit Formula* relating primes to zeros of the Euler-Riemann zeta function. The *idea* is that equating the Euler product and Riemann-Hadamard product for zeta allows extraction of an *exact formula* for a weighted counting of primes in terms of a sum over zeros of zeta. [1]

An essential supporting point is *meromorphic continuation* of $\zeta(s)$ via *integral representation(s)* of $\zeta(s)$ in terms of *theta function(s)*. [2] Further, these integral representations give *vertical growth estimates*, allowing invocation of Hadamard's theorem on product expansions of entire functions.

A key in analytic continuation and functional equation of $\zeta(s)$ is the functional equation of theta series, from the *Poisson summation formula*, from the representability of smooth functions by their *Fourier series*.

Asymptotics of $\Gamma(s)$ and the functional equation of $\zeta(s)$ bound the vertical growth of $\zeta(s)$, allowing application of the Hadamard product result.

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1. *Riemann's explicit formula*

The dramatic [Riemann 1859] on the relation between primes and zeros of the zeta function anticipated many ideas undeveloped in Riemann's time. Thus, the following sketch, very roughly following Riemann, is not a proof, but exhibits what is *needed* to produce a proof.

Riemann knew from Euler that $\zeta(s)$ has an *Euler product* expansion

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}} \qquad\qquad (\text{for } \mathrm{Re}\, s > 1)$$

As below, [Riemann 1859] proved that $\zeta(s)$ has a *meromorphic continuation* so that $(s-1)\zeta(s)$ is *entire*, with $0 = \zeta(0) = \zeta(-2) = \zeta(-4) = \ldots$. [3] The negative even integer are the *trivial* zeros of $\zeta(s)$. Riemann

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[1] [Guinand 1947] and [Weil 1952], [Weil 1972] observed that such classical formulas are equalities of values of a natural *distribution*, in the sense of *generalized functions*.

[2] Theta functions are examples of *automorphic forms*. For practical purposes, *modular form* and *automorphic form* are synonyms, despite some sources' attempts to insist upon delicately precise meanings.

[3] The vanishing at negative even integers is not clear at all, but will follow from the *functional equation*. Even so, Euler had already done computations with divergent series that could be interpreted as suggesting this!

imagined that $\zeta(s)$ has a *product expansion* in terms of its zeros [4]

$$(s-1)\,\zeta(s) \;=\; e^{a+bs}\prod_\rho \Big(1-\frac{s}{\rho}\Big)\,e^{s/\rho}\cdot \prod_{n=1}^\infty \Big(1+\frac{s}{2n}\Big)\,e^{-s/2n} \qquad (\rho \text{ non-trivial zero of } \zeta, \text{ for all } s\in\mathbb{C})$$

[Hadamard 1893] proved this. *Then*, taking logarithmic derivatives of

$$(s-1)\prod_p \frac{1}{1-\frac{1}{p^s}} \;=\; e^{a+bs}\prod_\rho \Big(1-\frac{s}{\rho}\Big)\,e^{s/\rho}\cdot \prod_{n=1}^\infty \Big(1+\frac{s}{2n}\Big)\,e^{-s/2n} \qquad (\mathrm{Re}\, s>1)$$

using $-\log(1-x)=x+x^2/2+x^3/3+\ldots$ on the left-hand side gives

$$\frac{1}{s-1} - \sum_{m\ge 1,\, p}\frac{\log p}{p^{ms}} \;=\; b \;+\; \sum_\rho \Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) \;+\; \sum_n \Big(\frac{1}{s+2n}-\frac{1}{2n}\Big)$$

A slight rearrangement:

$$\sum_{m\ge 1,\, p}\frac{\log p}{p^{ms}} \;=\; \frac{1}{s-1} - b \;-\; \sum_\rho \Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) \;-\; \sum_n \Big(\frac{1}{s+2n}-\frac{1}{2n}\Big) \qquad (\text{for } \mathrm{Re}\, s>1)$$

Diverging slightly from Riemann's original treatment, apply the Perron identity [5] (see Appendix)

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{Y^s}{s}\,ds \;=\; \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{Y^s}{s}\,ds \;=\; \begin{cases} 1 & (\text{for } Y>1) \\ 0 & (\text{for } 0<Y<1)\end{cases} \qquad (\text{for } \sigma>0)$$

to the log-derivative identity multiplied by X^s/s. Assuming legitimacy of application of the Perron identity *term-wise* to X^s/s times the left-hand side,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s}\sum_{m,p}\frac{\log p}{p^{ms}}\,ds \;=\; \sum_{m,p}\log p\cdot \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s\cdot p^{-ms}}{s}\,ds \;=\; \sum_{p^m<X}\log p$$

Assuming legitimacy of using *residues* term-wise to evaluate X^s/s times the right-hand side, with $\sigma>1$,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s}\cdot\Big(\frac{1}{s-1}-b\;-\;\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big)\;-\;\sum_n\Big(\frac{1}{s+2n}-\frac{1}{2n}\Big)\Big)\,ds$$

$$=\;(X-1)\;-\;b\;-\;\sum_\rho\Big(\frac{X^\rho}{\rho}+\frac{1}{-\rho}+\frac{1}{\rho}\Big)\;-\;\sum_n\Big(\frac{X^{-2n}}{-2n}+\frac{1}{2n}-\frac{1}{2n}\Big)\;=\;X\;-\;(b+1)\;-\;\sum_\rho\frac{X^\rho}{\rho}\;+\;\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

This gives [vonMangoldt 1893]'s reformulation of **Riemann's Explicit Formula:**

$$\sum_{p^m<X}\log p \;=\; X\;-\;(b+1)\;-\;\sum_\rho\frac{X^\rho}{\rho}\;+\;\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

[4] Euler's evaluation of $\sum_n \frac{1}{n^2}$ by imagining (and later proving) $\sin\pi z = \pi z\prod_n(1-\frac{z^2}{n^2})$ was well known, as was Euler's product expansion of the inverse of the Gamma-function $\Gamma(s)=\int_0^\infty t^s\,e^{-t}\,\frac{dt}{t}$ as $\frac{1}{\Gamma(s)}=ze^{\gamma z}\prod_n(1+\frac{z}{n})\,e^{-z/n}$.

[5] Perron's identity is completely standard by now, but was not part of Riemann's approach. Invocation of the Perron identity allows a somewhat simpler approach than Riemann's original, due to von Mangoldt and others.

More precisely, because of the way the Perron integral transform is applied, and the fragility of the convergence,

$$\sum_{p^m<X} \log p \;=\; X \;-\; (b+1) \;-\; \lim_{T\to\infty} \sum_{|\mathrm{Im}\,(\rho)|<T} \frac{X^\rho}{\rho} \;+\; \sum_{n\geq 1} \frac{X^{-2n}}{2n}$$

[1.0.1] Remark: As in Riemann's original, the above sketch has gaps. The existence and convergence of the Hadamard product needs *generalities* about Weierstraß-Hadamard product expressions for entire functions of prescribed growth, and *specifics* about the vertical growth of the *analytic continuation* of $\zeta(s)$. The analytic continuation of $\zeta(s)$ is discussed in the next section, and growth properties later. The growth properties depend on Stirling-Laplace asymptotics of the Gamma function $\Gamma(s)$, and the *Phragmén-Lindelöf* theorem [Phragmén-Lindelöf 1908].

[1.1] Non-trivial zeros ρ of $\zeta(s)$ The convergent Euler product shows that $\zeta(s) \neq 0$ in the half-plane $\mathrm{Re}\,(s) > 1$. The analytic continuation and functional equation (below), and relatively elementary properties of $\Gamma(s)$ show that the only possible non-trivial zeros are in the *critical strip* $0 \leq \mathrm{Re}\,(s) \leq 1$. In 1896, Hadamard and de la Vallée-Poussin independently proved that there are no zeros on the edges $\mathrm{Re}\,(s) = 0, 1$ of the critical strip, and used this to prove the *Prime Number Theorem.* The functional equation shows that if ρ is a non-trivial zero, then $1-\rho$ is a non-trivial zero. The property $\zeta(\bar{s}) = \overline{\zeta(s)}$ shows that if ρ is a non-trivial zero, then $\bar{\rho}$ is a non-trivial zero.

[1.2] The Riemann Hypothesis

After the main term X in the right-hand side of the explicit formula, the next-largest terms would be the X^ρ/ρ summands, with $0 \leq \mathrm{Re}\,(\rho) \leq 1$ due to the Euler product and functional equation. The *Riemann Hypothesis* is that all the non-trivial zeros ρ have $\mathrm{Re}\,(\rho) = \frac{1}{2}$. With a bound like $T\log T$ on the number of zeros below height T, proven later, the Riemann hypothesis is equivalent to an error term of order $X^{\frac{1}{2}+\varepsilon}$ in the Prime Number Theorem, for all $\varepsilon > 0$.

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2. *Analytic continuation and functional equation of $\zeta(s)$*

The following ideas gained publicity and significance from Riemann, but were apparently known earlier, to some degree.

The key is that the completed zeta function has an *integral representation* in terms of an *automorphic form*, the simplest *theta function.* Both the *analytic continuation* and the *functional equation* of zeta follow from this integral representation using a functional equation of the theta function, from *Poisson summation*, from *Fourier series.*

[2.1] Elementary-but-insufficiently-enlightening argument for analytic continuation Simple calculus can extend the domain of $\zeta(s)$ as far to the left as we want. The idea is to pay attention to *quantitative* aspects of the integral test. First, by comparison to $\int_1^\infty \frac{dx}{x^s}$, the sum $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$ converges for $\mathrm{Re}\,(s) > 1$.

To push this further, it is standard to proceed as follows.

$$\zeta(s) - \frac{1}{s-1} \;=\; \zeta(s) - \int_1^\infty \frac{dx}{x^s} \;=\; \sum_n \Big(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\Big) \;=\; \sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big)$$

Even for complex s, we have a Taylor-Maclaurin expansion with *error term*

$$(n+1)^{1-s} \;=\; \big(n\cdot(1+\frac{1}{n})\big)^{1-s} \;=\; n^{1-s}\cdot\big(1+\frac{1-s}{n} + O(\tfrac{1}{n^2})\big) \;=\; \frac{1}{n^{s-1}} - \frac{s-1}{n^s} + O(\frac{s-1}{n^{s+1}})$$

The constant in the big-O term is *uniform* in n for fixed s. Thus,

$$\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big] \;=\; \frac{1}{n^s} - \frac{1}{n^s} + \frac{1}{s-1}O\big(\frac{1}{n^{s+1}}\big) \;=\; O\big(\frac{1}{n^{s+1}}\big)$$

That is, for fixed [6] $\mathrm{Re}(s) > 0$, we have *absolute convergence* of

$$\sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big) \qquad\qquad (\text{for } \mathrm{Re}(s) > 0)$$

in the larger region $\mathrm{Re}(s) > 0$.

[2.1.1] Remark: Iterating the idea of approximating sums by integrals gives a comparable extension to $\mathrm{Re}(s) > -\ell$ for all ℓ, as Euler already effectively found, systematically by *Euler-Maclaurin summation*. However, such continuations give no clues about functional equations, and certainly not about Riemann's explicit formula.

[2.2] Slight modernization of Riemann's argument

We update Riemann's idea to avoid needless artifacts. Both the original and this update are archetypes. [7] Let $f(x)$ be *any* very well-behaved function on $\mathbb{R}$, that is, infinitely differentiable, and it and all its derivatives are rapidly decreasing at infinity. These are *Schwartz functions*, after [Schwartz 1950/51]. Further, take f *even*, that is $f(-x) = f(x)$. The even Schwartz function f is a *dummy*, insofar as only its general properties are used. In effect, Riemann's choice was the Gaussian $f(x) = e^{-\pi x^2}$, based on connections to Jacobi's theta functions, as we see along the way. A *theta function* [8] associated to the even Schwartz function f is

$$\theta_f(y) \;=\; \sum_{n\in\mathbb{Z}} f(y\cdot n) \qquad\qquad (\text{for } y > 0)$$

and associated *Gamma function* [9]

$$\Gamma_f(s) \;=\; \int_0^\infty t^s\, f(t)\, \frac{dt}{t}$$

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[6] In fact, the big-O constant is also *uniform* for s in compacts inside $\mathrm{Re}(s) > 0$. Thus, the series converges *locally uniformly on compacts*, so does give a *holomorphic* function.

[7] Riemann's original line of argument was brought to completion by [Hecke 1918/20]. Substantial modernization occurred in [Matchett 1946], [Iwasawa 1950/52], [Iwasawa 1952], and [Tate 1950/1967]. In particular, these sources observed that certain details involving *theta functions* were less essential than previously believed. Nevertheless, the *automorphic* nature of theta functions was *also* important in its own right.

[8] Again, Riemann used $f(u) = e^{-\pi u^2}$, and, consistent with an existing convention at the time, in effect defined

$$\theta(iy) \;=\; \sum_{n\in\mathbb{Z}} f(\sqrt{y}\cdot n) \qquad\qquad (\text{with Gaussian } f(u) = e^{-\pi u^2})$$

That is, the argument of θ is iy rather than y, and $\sqrt{y}$ enters on the right side, rather than y. Further, the Gaussian extends to an *entire* function, and this theta function extends to a holomorphic function, the simplest *Jacobi theta function*, on the upper half-plane $\mathfrak{H}$:

$$\theta(z) \;=\; \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad\qquad (\text{with } z\in\mathfrak{H})$$

[9] With Gaussian $f(x) = e^{-\pi x^2}$, this construction gives an exponential multiple of the standard Gamma function at $\frac{s}{2}$:

$$\Gamma_f(s) \;=\; \int_0^\infty t^s\, e^{-\pi x^2}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\int_0^\infty t^{\frac{s}{2}}\, e^{-\pi x}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\,\pi^{-\frac{s}{2}}\int_0^\infty t^{\frac{s}{2}}\, e^{-x}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\,\pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})$$

First, we have the *integral representation*, from which will follow the meromorphic continuation and functional equation:

[2.2.1] Proposition: $\displaystyle\int_0^\infty y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y} \;=\; \Gamma_f(s) \cdot \zeta(s)$ (for $\mathrm{Re}(s) > 1$)

Proof: The $n = 0$ (constant) term $f(0)$ of $\theta_f(y)$ is the only summand not rapidly decreasing. The even-ness of f makes the $\pm n$ terms have equal contributions to $\theta_f(y)$. Thus, interchanging sum and integral, and replacing y by y/n,

$$\int_0^\infty y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y} \;=\; \sum_{n\geq 1} \int_0^\infty y^s \; f(yn) \; \frac{dy}{y} \;=\; \sum_{n\geq 1} n^{-s} \int_0^\infty y^s \; f(y) \; \frac{dy}{y} \;=\; \sum_{n\geq 1} n^{-s} \, \Gamma_f(s)$$

as claimed. ///

[2.2.2] Remark: The measure $\frac{dy}{y}$ is the natural multiplication-invariant measure on the positive reals.

[2.2.3] Theorem: The completed zeta function $\Gamma_f(s) \cdot \zeta(s)$ has a meromorphic continuation to $s \in \mathbb{C}$, and $s(s-1) \cdot \Gamma_f(s) \cdot \zeta(s)$ is *entire.*

[2.2.4] Remark: Repeated integration by parts shows that $\Gamma_f(s)$ itself has a meromorphic continuation:

$$\Gamma_f(s) \;=\; \int_0^\infty t^s \, f(t) \, \frac{dt}{t} = \int_0^\infty \frac{t^{s+1}}{s} \, f'(t) \, \frac{dt}{t} = \int_0^\infty \frac{t^{s+2}}{s(s+1)} \, f''(t) \, \frac{dt}{t} = \int_0^\infty \frac{t^{s+3}}{s(s+1)(s+2)} \, f'''(t) \, \frac{dt}{t} = \ldots$$

Since all the derivatives of f are of rapid decay, these expressions give an extension of $\Gamma_f(s)$ to $s \in \mathbb{C}$ except for at worst $s = 0, -1, -2, -3, \ldots$.

Proof: Break the integral of the integral representation into two parts:

$$\Gamma_f(s) \cdot \zeta(s) \;=\; \int_1^\infty y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y} \;+\; \int_0^1 y^s \; \frac{\theta_f(y) - f(0)}{2} \; \frac{dy}{y}$$

It is not hard to check that $\frac{\theta_f(y) - \theta_f(0)}{2}$ is rapidly decreasing at $+\infty$, so the integral on $[1, \infty)$ is absolutely convergent (and uniformly for s in compacts) for all $s \in \mathbb{C}$.

The behavior of $\theta_f(y)$ as $y \to 0^+$ is harder to analyze, and is best done by the following device.

The trick is to convert the integral on $[0, 1]$ to an integral over $[1, \infty)$, up to two elementary terms. The new integral over $[1, \infty)$ will involve the theta function $\theta_{\widehat{f}}$ attached to the *Fourier transform*

$$\widehat{f}(x) \;=\; \int_{\mathbb{R}} e^{-2\pi i x \xi} \; f(\xi) \, d\xi$$

of f. We grant for the moment that Fourier transform maps the Schwartz space to itself, as is directly verifiable in concrete examples such as the Gaussian $f(x) = e^{-\pi x^2}$. Simply by changing variables in the integral, we recall a homogeneity property of the Fourier transform:

$$\widehat{f}(x/y) \;=\; \int_{\mathbb{R}} e^{-2\pi i \frac{x}{y} \xi} \; f(\xi) \, d\xi \;=\; |y| \int_{\mathbb{R}} e^{-2\pi i x \xi} \; f(y\xi) \, d\xi \;=\; |y| \cdot (f \circ y)\widehat{\;}(x)$$

by replacing ξ by ξy in the integral, where $(f \circ y)(\xi) = f(y\xi)$. We grant ourselves the standard *Poisson summation formula*

$$\sum_{n \in \mathbb{Z}} F(n) \;=\; \sum_{n \in \mathbb{Z}} \widehat{F}(n) \qquad \text{(for Schwartz functions } F\text{)}$$

(See the Supplement for proof.) Letting $F(x) = f(yx)$ and using the homogeneity property of Fourier transform, this is

$$\sum_{n\in\mathbb{Z}} f(y\cdot n) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{y}\,\widehat{f}\Big(\frac{1}{y}\cdot n\Big) \qquad\qquad \text{(for } y>0\text{)}$$

Thus,

$$\theta_f(y) \;=\; \sum_{n\in\mathbb{Z}} f(yn) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{y}\widehat{f}(n) \;=\; \frac{1}{y}\cdot\theta_{\widehat{f}}\Big(\frac{1}{y}\Big)$$

This gives a way to flip the interval $[0,1]$ to $[1,\infty)$, by replacing y by $1/y$, accommodating the anomalous terms for $n=0$ separately:

$$\int_0^1 y^s\,\frac{\theta_f(y)-f(0)}{2}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta_{\widehat{f}}(\frac{1}{y})-f(0)}{2}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta_{\widehat{f}}(\frac{1}{y})-\frac{1}{y}\widehat{f}(0)}{2} + \frac{\frac{1}{y}\widehat{f}(0)-f(0)}{2}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{-s}\,\frac{y\theta_{\widehat{f}}(y)-y\widehat{f}(0)}{2} + \int_0^1 y^s\,\frac{\frac{1}{y}\widehat{f}(0)-f(0)}{2}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\,\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\int_0^1 y^{s-1}\,\frac{dy}{y} \;-\; \frac{f(0)}{2}\int_0^1 y^s\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\,\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

The integral on $[1,\infty)$ is entire in s, since $\theta_{\widehat{f}}(y)-\widehat{f}(0)$ is rapidly decreasing at ∞. The two elementary terms have obvious meromorphic continuations. Thus,

$$\Gamma_f(s)\cdot\zeta(s) \;=\; \int_1^\infty \Big(y^s\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2} + y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\Big)\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

Again, the integral is *entire*, and the elementary terms give the only poles, which are at $s=0,1$. ///

[2.2.5] Remark: The expression

$$\Gamma_f(s)\cdot\zeta(s) \;=\; \int_1^\infty \Big(y^s\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2} + y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\Big)\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

gives a bit more information than the bare statement of the theorem, namely, it tells the residues of the poles at $s=0,1$, and shows a certain potential symmetry, as in the following.

For f with $\widehat{f}=f$ Riemann's original symmetrical result is recovered:

[2.2.6] Theorem: *(Riemann)* The *completed* zeta function

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

has an analytic continuation to $s\in\mathbb{C}$, except for simple poles at $s=0,1$, and has the *functional equation*

$$\xi(1-s) \;=\; \xi(s)$$

Proof: Various means show that $f(x)=e^{-\pi x^2}$ is its own Fourier transform. Thus, the expression in the proof of the previous theorem becomes symmetrical in $s\leftrightarrow 1-s$, and the artifact of the coefficient of $\frac{1}{2}$ on both sides can be discarded. ///

[2.2.7] Remark: The leading factor $\pi^{-s/2}\Gamma(\frac{s}{2})$ should *not* be construed as objectionable in any way, but, rather, as something that really does *belong* with $\zeta(s)$. The $\pi^{-s/2}\Gamma(\frac{s}{2})$ is called the **gamma factor** for $\zeta(s)$. In the context of the *Euler product* the modern viewpoint is that the gamma factor is a further Euler factor corresponding to the *prime*[10] ∞.

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3. *Appendix: Perron identity*

These contour-integral identities extract information from spectral identities and function-theoretic identities. *One* spectral identity is transformed into *another*, by a Fourier transform. Choices are made to heighten an *asymmetry*, wherein one side is seemingly elementary, and the other is whatever it must be.

[3.1] Heuristic The best-known identity starts from the *idea* that for $\sigma > 0$

$$\int_{\sigma-i\infty}^{\sigma-i\infty} \frac{X^s}{s}\, ds \;=\; \begin{cases} 1 & (\text{for } X > 1) \\ \\ 0 & (\text{for } 0 < X < 1) \end{cases} \qquad\qquad (\text{convergence?})$$

The *idea* of the proof of this identity is that, for $X > 1$, the contour of integration slides indefinitely to the left, eventually vanishing, picking up the residue at $s = 0$, while for $0 < X < 1$, the countour slides indefinitely to the right, eventually vanishing, picking up *no* residues.

The *idea* of the application is that this identity can extract *counting* information from a meromorphic continuation of a Dirichlet series: for example, from

$$\sum_n \frac{a_n}{n^s} \;=\; f(s) \qquad\qquad (\text{left-hand side convergent for } \mathrm{Re}\, s > 1)$$

we would have

$$\sum_{n<X} a_n \;=\; \text{sum of residues of } X^s\, f(s)/s$$

That is, the *counting* function $\sum_{n<X} a_n$ is *extracted* from the analytic object $\sum_\lambda a_n/n^s$ by the contour integration. With f a logarithmic derivative, such as $f(s) = \zeta'(s)/\zeta(s)$, the poles of f are mostly the zeros of ζ.

However, the tails of these integrals are fragile.

[3.2] Simple precise assertion The elegant simplicity of the idea about moving lines of integration must be elaborated for correctness: for fixed $\sigma > 0$, for $T > 0$, we claim that

$$\int_{\sigma-iT}^{\sigma-iT} \frac{X^s}{s}\, ds \;=\; \begin{cases} 1 + O_\sigma(\frac{X^\sigma}{T\cdot \log X}) & (\text{for } X > 1) \\ \\ O_\sigma(\frac{X^\sigma}{T\cdot |\log X|}) & (\text{for } 0 < X < 1) \end{cases}$$

The proof is a precise form of the idea of sliding vertical contours. That is, for $X > 1$, consider the contour integral around the rectangle with *right* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $-B + iT$, $-B - iT$, with $B \to +\infty$. For $0 < X < 1$ consider the contour integral around the rectangle with *left* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $B + iT$, $B - iT$, with $B \to +\infty$.

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[10] An insight of modern times is that the completion $\mathbb{R}$ should whenever possible be put on an even footing with the other p-adic completions $\mathbb{Q}_p$ of $\mathbb{Q}$. Thus, although there is no actual prime ∞ in $\mathbb{Z}$ (or anywhere else), the objects that accompany genuine primes p and completions $\mathbb{Q}_p$ often have analogues for $\mathbb{R}$, so we *backform* to refer to the *prime* ∞. One attempt to be less bold in this regard is to speak of *places* rather than *primes*, but there's little point in fretting about this.

For both $X > 1$ and $0 < X < 1$, the $\pm(B \pm iT)$ edge of the rectangle is dominated by

$$\int_{-T}^{T} \frac{e^{-B|\log X|}}{|B \pm it|}\,dt \;\ll\; T \cdot \frac{e^{-B|\log X|}}{B} \;\to\; 0 \qquad\qquad (\text{as } B \to +\infty)$$

in both cases, the top and bottom edges of the rectangle are dominated by

$$X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{|(\sigma \pm u) + iT|}\,du \;\ll\; X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{T}\,du \;\ll\; \frac{X^\sigma}{T \cdot |\log X|}$$

This proves the claim. Replacing X by e^X in the estimate gives the equivalent

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} \frac{e^{sX}}{s}\,ds \;=\; \begin{cases} 1 + O_\sigma(\frac{e^{\sigma X}}{T \cdot X}) & (\text{for } X > 0) \\ O_\sigma(\frac{e^{\sigma X}}{T \cdot |X|}) & (\text{for } X < 0) \end{cases}$$

[3.3] Hazards

When the quantity X above is summed, especially if the summation is over a set whose precise specifications are difficult, the denominators of the big-O error terms may blow up. In situations such as

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma + iT} \Big(\sum_j a_j\, e^{-sX_j}\Big)\frac{e^{sX}}{s}\,ds \;=\; \sum_{j\,:\,X_j < X} a_j \;+\; \sum_j a_j \cdot O_\sigma\Big(\frac{e^{\sigma(X - X_j)}}{T \cdot |X - X_j|}\Big)$$

the distribution of the values X_j has an obvious effect on the convergence of the error term.

[3.4] The other side of the equation

A desired and plausible conclusion such as

$$\lim_T \frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} f(s)\,\frac{e^{sX}}{s}\,ds \;=\; \big(\text{sum of } \mathrm{Res}_{s=\rho}\, f(s) \cdot \tfrac{e^{\rho X}}{\rho}\big)$$

summed over poles ρ of f in the left half-plane $\mathrm{Re}\, s < \sigma$, requires that the contour integrals over the other three sides of the rectangle with side $\sigma \pm iT$ go to 0, and that the tails of the vertical integral go to 0. The integral over the large rectangle will be evaluated with X large positive, so the decay condition applies to f to the *left*. The left side of the rectangle will go to 0 for large enough positive X when $f(s)$ has at worst exponential growth to the left, that is, when $f(s) \ll e^{-C \cdot |\mathrm{Re}\, s|}$ for *some* large-enough C and $\mathrm{Re}\, s \to -\infty$. The top and bottom are more fragile, since e^{sX}/s does not have strong decay vertically.

Not unexpectedly, the *poles* of f near $\sigma + iT$ may *bunch up* as T grows, so that a countour integral must be **threaded** between them, and the corresponding integral will be somewhat larger simply because of proximity to these poles. This contribution to vertical growth of f is significant in examples.

[3.5] Variant identities

When X^s/s is altered to help convergence of the integral against the *counting* aspect is inevitably altered. The proofs of variants follow the same straightforward line as above for the simplest case. For $\theta > 0$ and $1 \le \ell \in \mathbb{Z}$,

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} \frac{X^s}{s(s+\theta)(s+2\theta)\ldots(s+\ell\theta)}\,ds \;=\; \begin{cases} \frac{1}{\ell!\theta^\ell}(1 - X^{-\theta})^\ell + O_\sigma(\frac{X^\sigma}{T^2 \cdot \log X}) & (\text{for } X > 1) \\ O_\sigma(\frac{X^\sigma}{T^2 \cdot |\log X|}) & (\text{for } 0 < X < 1) \end{cases}$$

Indeed, the residues at the poles $0, -\theta, -2\theta, \ldots, -\ell\theta$ sum to

$$\frac{X^0}{(0+\theta)(0+2\theta)\cdots(0+(\ell-1)\theta)(0+\ell\theta)} + \frac{X^{-\theta}}{(-\theta+0)(-2\theta+\theta)\cdots(-\theta+(\ell-1)\theta)(-\theta+\ell\theta)}$$

$$+ \frac{X^{-2\theta}}{(-2\theta+0)(-2\theta+\theta)\cdots(-2\theta+\ell\theta)} + \ldots + \frac{X^{-\ell\theta}}{(-\ell\theta+0)(-\ell\theta+\theta)\cdots(-\ell\theta+(\ell-1)\theta)}$$

$$= \frac{1}{\ell!\,\theta^\ell} - \frac{X^{-\theta}}{1!\,(\ell-1)!\,\theta^\ell} + \frac{X^{-2\theta}}{2!\,(\ell-2)!\,\theta^\ell} + \ldots \pm \frac{X^{-\ell\theta}}{\ell!\,0!\,\theta^\ell} \;=\; \frac{(1 - X^{-\theta})^\ell}{\ell!\,\theta^\ell}$$

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