

Guinand's explicit formula

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We take for granted the basic analytical properties of Riemann's zeta function and the Gamma function. In particular, with $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$ the Gamma factor for $\zeta(s)$, and $\xi(s) = \Gamma_{\mathbb{R}}(s) \cdot \zeta(s)$, the functional equation is $\xi(1-s) = \xi(s)$. The completed zeta function ξ has poles only at $s = 0, 1$, and these simple. We use the normalization of Fourier transform [1]

$$\widehat{g}(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x) dx$$

[0.0.1] **Theorem:** (*Guinand*) For $g \in C_c^\infty(\mathbb{R})$, letting $\rho = \frac{1}{2} + i\gamma$ whether or not ρ is on the critical line,

$$\begin{aligned} & \sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\rho) - \widehat{g}(i/2) - \widehat{g}(-i/2) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}} \left(\frac{1}{2} + it \right) + \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}} \left(\frac{1}{2} - it \right) \right) \widehat{g}(t) dt - \sum_{p^m} \frac{\log p}{\sqrt{p^m}} \left(g(m \log p) + g(-m \log p) \right) \end{aligned}$$

Proof: The idea is in [Guinand 1947], and there is essentially only one route to take. [2] Consider

$$I = \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=1+\varepsilon} \frac{\xi'(s)}{\xi(s)} \widehat{g}(t) ds \quad (\text{with } s = \frac{1}{2} + it \text{ whether or not } \operatorname{Re}(s) = \frac{1}{2})$$

The integral makes sense because $\widehat{g}(s)$ is *entire*. It converges because \widehat{g} is rapidly decreasing in $\operatorname{Re}(s)$, for every fixed $\operatorname{Im}(s)$. The logarithmic derivative has simple poles at the zeros and poles of $\xi(s)$ with residues the order of vanishing, namely, the two simple poles at $s = 0, 1$, trivial zeros at $s = -2, -4, -6, \dots$, and non-trivial zeros ρ in the critical strip. In terms of $t = (s - \frac{1}{2})/i$, these are at $\pm i/2$. Moving the contour [3] to $\operatorname{Re}(s) = -\varepsilon$ captures $t = \pm i/2$ and the non-trivial zeros ρ , written similarly as $\rho = \frac{1}{2} + i\gamma$, whether or not we know that $\operatorname{Re}(\gamma) = \frac{1}{2}$:

$$I = -\widehat{g}(i/2) - \widehat{g}(-i/2) + \sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\rho) + \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=-\varepsilon} \frac{\xi'(s)}{\xi(s)} \widehat{g}(t) ds$$

From the functional equation, $\frac{\xi'}{\xi}(s) = -\frac{\xi'}{\xi}(1-s)$, and the integral on $\operatorname{Re}(s) = -\varepsilon$ is equal to an integral on $\operatorname{Re}(s) = 1 + \varepsilon$:

$$I = -\widehat{g}(i/2) - \widehat{g}(-i/2) + \sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\rho) - \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=1+\varepsilon} \frac{\xi'(1-s)}{\xi(1-s)} \widehat{g}(t) ds$$

[1] Often, there might be a 2π somewhere, or a different sign in the exponent, but those details are irrelevant. The present choice is for convenience in the situation at hand.

[2] Compare [Weil 1952/1972], [Rudnick-Sarnak 1996] page 277 ff., or [Iwaniec-Kowalski 2004] page 508 ff.

[3] To legitimize this contour move, we should identify a sequence $T_j \rightarrow +\infty$ such that the integrals along $[-\varepsilon + iT_j, 1 + \varepsilon + iT_j]$ and $[-\varepsilon - iT_j, 1 + \varepsilon - iT_j]$ go to 0. Since $\widehat{g}(t)$ is rapidly decreasing, this is not delicate. However, it still does need an asymptotic estimate of the number of zeros of $\zeta(s)$ to height T , which comes from the functional equation, Stirling-Laplace asymptotics for $\Gamma(s)$, and Hadamard's theorem on growth versus zeros of entire functions.

Rearranging,

$$\sum_{\rho=\frac{1}{2}+i\gamma} \widehat{g}(\rho) - \widehat{g}(i/2) - \widehat{g}(-i/2) = \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=1+\varepsilon} \frac{\xi'(s)}{\xi(s)} \widehat{g}(t) ds + \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=1+\varepsilon} \frac{\xi'(1-s)}{\xi(1-s)} \widehat{g}(t) ds$$

Separating the archimedean factors,

$$\frac{\xi'(s)}{\xi(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)}$$

The contour integrals of the archimedean part can be shifted to $\operatorname{Re}(s) = \frac{1}{2}$ without picking up any residues, giving

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\frac{1}{2}} \frac{\Gamma'_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s)} \widehat{g}(t) ds + \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=\frac{1}{2}} \frac{\Gamma'_{\mathbb{R}}(1-s)}{\Gamma_{\mathbb{R}}(1-s)} \widehat{g}(t) ds = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(t) \left(\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}} \left(\frac{1}{2} + it \right) + \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}} \left(\frac{1}{2} - it \right) \right) dt$$

The logarithmic derivative of the finite-prime part, in a right half-plane, is [4]

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_{p^m \geq 1} \frac{\log p}{p^{ms}}$$

giving

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(s)=1+\varepsilon} \frac{\zeta'(s)}{\zeta(s)} \widehat{g}(t) ds = - \sum_{p,m} \log(p) \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=1+\varepsilon} \frac{1}{p^{ms}} \cdot \widehat{g}(t) ds$$

Moving the contour to $\operatorname{Re}(s) = \frac{1}{2}$ gives $p^{ms} = \sqrt{p^m} \cdot e^{itm \log p}$, and with $ds = d(\frac{1}{2} + it) = i dt$, the integral becomes

$$- \sum_{p,m} \frac{\log p}{\sqrt{p^m}} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itm \log p} \widehat{g}(t) dt$$

Here, Fourier inversion is

$$\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(t) e^{-itx} dt = g(x)$$

Thus, by Fourier inversion,

$$- \sum_{p,m} \log p \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{p^m}} \cdot e^{-itm \log p} \widehat{g}(t) dt = - \sum_{p,m} \frac{\log p}{\sqrt{p^m}} g(m \log p)$$

The $1-s$ term is treated similarly, giving the assertion of the theorem. ///

[Guinand 1947] A. P. Guinand, *Some Fourier transforms in prime-number theory*, Quart. J. Math., Oxford **18** (1947), 53-64.

[Iwaniec-Kowalski 2004] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS Coll. Publ., 2004.

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[4] A different notational choice is to sum over all integers, rather than over powers of primes, and use the *von Mangoldt* function

$$\Lambda(n) = \begin{cases} \log p & (\text{for } n = p^k, \text{ prime } p, 1 \leq k \in \mathbb{Z}) \\ 0 & (\text{otherwise}) \end{cases}$$

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