Infinitude of zeros in the critical strip

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

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Hadamard’s theorem on canonical products yields a short proof that \( \zeta(s) \) has infinitely-many zeros in the critical strip \( 0 \leq \text{Re}(s) \leq 1 \). This is essentially an echo of [Titchmarsh 1986], page 30, and some background.

Hadamard’s product theorem, for growth order \( \lambda \in \mathbb{R} \), asserts that for the integer \( h \) satisfying \( h \leq \lambda < h+1 \), an entire function \( f \) of order \( \lambda \) has product expansion

\[
f(z) = e^{g(z)} \cdot z^\nu \prod_{z_i} \left( 1 - \frac{z}{z_i} \right) e^{p_h(z/z_i)}
\]

where \( \nu \) is the order of 0 at 0, \( z_i \) runs through non-zero zeros of \( f \), \( g(z) \) is a polynomial of degree at most \( h \), and \( p_h(z) \) is the \( h^{th} \) truncation of the Taylor series for \( \log(1-z) \), namely,

\[
p_h(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^h}{h}
\]

For \( h = 0 \), take \( p_0(z) = 0 \). Hadamard’s theorem controls the leading exponential: rather than being \( e^{g(z)} \) with some unfathomable entire function \( g(z) \), we have sharp constraints on \( g(z) \).

Thus, there is the peculiar corollary that entire functions of growth order \( \lambda < 1 \) have \( h = 0 \), so have very simple product expansions

\[
f(z) = e^a \cdot z^\nu \prod_{z_i} \left( 1 - \frac{z}{z_i} \right) \quad \text{(for } f \text{ entire of order } \lambda < 1\text{)}
\]

for some constant \( a \). In particular, if \( f \) is not a polynomial, then it has infinitely-many zeros.

This corollary can be used to prove that \( \zeta(s) \) has infinitely-many zeros in the strip \( 0 \leq \text{Re}(s) \leq 1 \), as follows.

From the functional equation, and from the fact that \( \Gamma(s) \) has no zeros, the only possible zeros of \( \xi(s) \) are in \( 0 \leq \text{Re}(s) \leq 1 \).

Let \( \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) \). In light of the functional equation \( \xi(1-s) = \xi(s) \) and the fact that \( \xi(s) \) has exactly two poles, at \( s = 0, 1 \), which are simple, the function \( s(1-s)\xi(s) \) is entire and still satisfies the same equation. That is \( z \to (\frac{1}{2} + z)(\frac{1}{2} - z)\xi(\frac{1}{2} + z) \) is entire and even. Thus, it is a function of \( z^2 \), and there is an entire function \( f \) such that

\[
f(z^2) = (\frac{1}{2} + z)(\frac{1}{2} - z)\xi(\frac{1}{2} + z)
\]

There is a traditionally-defined function \( \Xi(z) \) which differs from this \( f \) only in normalization. We have shown that \( \xi(s) \) is of growth-order 1, so \( f \) is of growth-order \( \frac{1}{2} \). Thus, by the corollary to Hadamard’s theorem, either \( f \) is a polynomial, or has infinitely-many zeros. If \( f(z) \) were a polynomial, then \( f(z^2) \) would be a polynomial, as well. But the super-polynomial growth of \( \pi^{-s/2} \Gamma(s/2)\zeta(s) \) for \( s \) real and \( s \to +\infty \) shows that this is impossible. Thus, \( f \) has infinitely-many zeros.