(*March 11, 2015*)

Discriminant of cubics

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/10b_cubic_discriminant.pdf]

We give a reproducible proof for the expression of the discriminant of a cubic in terms of the elementary symmetric polynomials:

[0.0.1] Claim: Given $(x-\alpha)(x-\beta)(x-\gamma) = x^3 - s_1x^2 + s_2x - s_3$, the *discriminant* is expressible as

$$\Delta \;=\; (\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 \;=\; (s_1^2-4s_2)s_2^2 + s_3(-4s_1^3+18s_1s_2-27s_3)$$

Proof: Imitating part of the proof that every symmetric polynomial is expressible in terms of the elementary ones, first set $\gamma = 0$, and observe that the discriminant degenerates into

$$(\alpha-\beta)^2\alpha^2\beta^2 \;=\; (\overline{s}_1^2 - 4\overline{s}_2)\overline{s}_2^2$$

where the $\overline{s}_j$ are the corresonding elementary symmetric functions $\overline{s}_1 = \alpha+\beta$ and $\overline{s}_2 = \alpha\beta$. Then $\Delta - (s_1^2-4s_2)s_2^2$ is symmetric and vanishes at $\gamma = 0$, thus, vanishes at $s_3 = 0$. Since $\mathbb{Z}[s_1, s_2, s_3]$ is isomorphic to a polynomial ring in three variables, it is a unique factorization domain, by Gauss and Eisenstein. Thus, s_3 *divides* $\Delta - (s_1^2-4s_2)s_2^2$.

The *homogeneity* properties

$$t\alpha+t\beta+t\gamma = t\cdot(\alpha+\beta+\gamma) \qquad (t\alpha)(t\beta)+(t\alpha)(t\gamma)+(t\beta)(t\gamma) = t^2\cdot(\alpha\beta+\alpha\gamma+\beta\gamma) \qquad (t\alpha)(t\beta)(t\gamma) = t^3\cdot\alpha\beta\gamma$$

of the elementary symmetric functions and of Δ shows that for some constants a, b, c

$$\frac{\Delta - (s_1^2-4s_2)s_2^2}{s_3} \;=\; as_1^3 + bs_1s_2 + cs_3$$

that is,

$$\Delta \;=\; (s_1^2-4s_2)s_2^2 + (as_1^3 + bs_1s_2 + cs_3)s_3$$

Successive simple choices of α, β, γ give linear equations solvable for a, b, c.

First, $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$ for cube root of unity ω conveniently makes $s_1 = s_2 = 0$ and $s_3 = 1$, so the expression for Δ directly gives

$$c \;=\; (1-\omega)^2(1-\omega^2)^2(\omega-\omega^2)^2 \;=\; -27$$

Second, $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ gives $s_1 = s_2 = 3$ and $s_3 = 1$, while $\Delta = 0$. Thus,

$$0 \;=\; (s_1^2-4s_2)s_2^2 + (as_1^3 + bs_1s_2 + cs_3)s_3 \;=\; (3^2 - 4\cdot 3)3^2 + (a\cdot 3^3 + b\cdot 3\cdot 3 - 27)$$

Dividing through by 9 and rearranging,

$$6 \;=\; 3a + b$$

Third, to get another linear relation, use $(x^2+1)(x-1) = x^3 - x^2 + x - 1$, so $s_1 = s_2 = s_3 = 1$. The discriminant is $\Delta = (1-i)^2(1+i)^2(i+i)^2 = -16$, so

$$-16 \;=\; (s_1^2-4s_2)s_2^2 + (as_1^3 + bs_1s_2 + cs_3)s_3 \;=\; (1-4) + (a+b-27)$$

or

$$14 \;=\; a + b$$

Substituting $b = 14 - a$ into $6 = 3a + b$ gives $6 = 3a + 14 - a$, so $-8 = 2a$, and $a = -4$. Thus, $b = 14 - (-4) = 18$, and

$$\Delta \;=\; (s_1^2 - 4s_2)s_2^2 + (-4s_1^3 + 18s_1 s_2 - 27s_3)s_3$$

///

[0.0.2] Remark: Another convenient data point is $(x^2 - 1)(x - 1) = x^3 - x^2 - x + 1$, with $s_1 = 1$, $s_2 = s_3 = -1$, to provide a check: the discriminant is 0, so we test whether or not

$$0 \;=\; (1 - 4(-1)) + (-4 + 18(-1) - 27(-1))(-1) \qquad\qquad (?)$$

Indeed,

$$(1 - 4(-1)) + (-4 + 18(-1) - 27(-1))(-1) \;=\; 5 - (-4 - 18 + 27) \;=\; 5 + 4 + 18 - 27 \;=\; 0$$

as hoped.
