Discriminant of cubics

Given \((x - \alpha)(x - \beta)(x - \gamma) = x^3 - s_1x^2 + s_2x - s_3\), the discriminant is expressible as

\[
\Delta = (\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2 = (s_1^2 - 4s_2)s_2^2 + s_3(-4s_1^3 + 18s_1s_2 - 27s_3)
\]

**Proof:** Imitating part of the proof that every symmetric polynomial is expressible in terms of the elementary ones, first set \(\gamma = 0\), and observe that the discriminant degenerates into

\[
(\alpha - \beta)^2\alpha^2\beta^2 = (\pi_1^2 - 4\pi_2)\pi_2^2
\]

where the \(\pi_j\) are the corresponding elementary symmetric functions \(\pi_1 = \alpha + \beta\) and \(\pi_2 = \alpha\beta\). Then \(\Delta - (s_1^2 - 4s_2)s_2^2\) is symmetric and vanishes at \(\gamma = 0\), thus, vanishes at \(s_3 = 0\). Since \(\mathbb{Z}[s_1, s_2, s_3]\) is isomorphic to a polynomial ring in three variables, it is a unique factorization domain, by Gauss and Eisenstein. Thus, \(s_3\) divides \(\Delta - (s_1^2 - 4s_2)s_2^2\).

The homogeneity properties

\[
t\alpha + \beta + \gamma = t(\alpha + \beta + \gamma) \quad (t\alpha)(t\beta) + (t\alpha)(t\gamma) + (t\beta)(t\gamma) = t^2(\alpha\beta + \alpha\gamma + \beta\gamma) \quad (t\alpha)(t\beta)(t\gamma) = t^3\alpha\beta\gamma
\]

of the elementary symmetric functions and of \(\Delta\) shows that for some constants \(a, b, c\)

\[
\frac{\Delta - (s_1^2 - 4s_2)s_2^2}{s_3} = as_1^3 + bs_1s_2 + cs_3
\]

that is,

\[
\Delta = (s_1^2 - 4s_2)s_2^2 + (as_1^3 + bs_1s_2 + cs_3)s_3
\]

Successive simple choices of \(\alpha, \beta, \gamma\) give linear equations solvable for \(a, b, c\).

First, \(x^3 - 1 = (x - 1)(x - \omega)(x - \omega^2)\) for cube root of unity \(\omega\) conveniently makes \(s_1 = s_2 = 0\) and \(s_3 = 1\), so the expression for \(\Delta\) directly gives

\[
c = (1 - \omega)^2(1 - \omega^2)^2(\omega - \omega^2)^2 = -27
\]

Second, \((x - 1)^3 = x^3 - 3x^2 + 3x - 1\) gives \(s_1 = s_2 = 3\) and \(s_3 = 1\), while \(\Delta = 0\). Thus,

\[
0 = (s_1^2 - 4s_2)s_2^2 + (as_1^3 + bs_1s_2 + cs_3)s_3 = (3^2 - 4 \cdot 3)3^2 + (a \cdot 3^3 + b \cdot 3 \cdot 3 - 27)
\]

Dividing through by 9 and rearranging,

\[
6 = 3a + b
\]

Third, to get another linear relation, use \((x^2 + 1)(x - 1) = x^3 - x^2 + x - 1\), so \(s_1 = s_2 = s_3 = 1\). The discriminant is \(\Delta = (1 - i)^2(1 + i)^2(i + i)^2 = -16\), so

\[
-16 = (s_1^2 - 4s_2)s_2^2 + (as_1^3 + bs_1s_2 + cs_3)s_3 = (1 - 4) + (a + b - 27)
\]
or

\[ 14 = a + b \]

Substituting \( b = 14 - a \) into \( 6 = 3a + b \) gives \( 6 = 3a + 14 - a \), so \( -8 = 2a \), and \( a = -4 \). Thus, \( b = 14 - (-4) = 18 \), and

\[ \Delta = (s_1^2 - 4s_2)s_2^2 + (-4s_1^3 + 18s_1s_2 - 27s_3)s_3 \]

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[0.0.2] **Remark:** Another convenient data point is \((x^2-1)(x-1) = x^3 - x^2 - x + 1\), with \( s_1 = 1 \), \( s_2 = s_3 = -1 \), to provide a check: the discriminant is 0, so we test whether or not

\[ 0 = (1 - 4(-1)) + (-4 + 18(-1) - 27(-1))(-1) \]

Indeed,

\[ (1 - 4(-1)) + (-4 + 18(-1) - 27(-1))(-1) = 5 - (-4 - 18 + 27) = 5 + 4 + 18 - 27 = 0 \]

as hoped.