(*March 24, 2015*)

The product expansion $\Delta(z) = e^{2\pi iz} \prod(1 - e^{2\pi inz})^{24}$

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/10e_product_expansion.pdf]

[Siegel 1954] showed a very simple, if somewhat unmotivated, argument for the product expansion of $\Delta(z)$. [Weil 1968] reproved the product expansion by a more complicated method related to the *converse theorems* in [Weil 1967], the latter arising as plausibility checks on the Taniyama-Shimura conjecture.

We reproduce both arguments. As expected, both make heavy use of various coincidences. Both use the one-dimensionality of the space of holomorphic cusp forms of weight 12 for $SL_2(\mathbb{Z})$ and generation of $SL_2(\mathbb{Z})$ by translation $z \to z+1$ and inversion $z \to -1/z$. A function of $e^{2\pi iz}$ is clearly invariant under $z \to z+1$. It remains to prove that the product expression must be proven to have the functional equation of a weight 12 modular form under $z \to -1/z$.

---

1. *Siegel's proof*

Siegel's argument is simple but *ad-hoc*. With η the 24^{th} root of Δ, with $q = e^{2\pi iz}$, taking a logarithm,

$$\frac{1}{12}\pi iz - \log\eta(z) \;=\; -\sum_{\ell=1}^{\infty}\log(1-q^\ell) \;=\; \sum_{k,\ell\ge 1}\frac{1}{k}q^{k\ell} \;=\; \sum_{k\ge 1}\frac{1}{k}\cdot\frac{q^k}{1-q^k} \;=\; \sum_{k\ge 1}\frac{1}{k}\cdot\frac{1}{q^{-k}-1}$$

This suggests proving the functional equation in the form

$$\pi i\frac{z+z^{-1}}{12} + \tfrac{1}{2}\log\frac{z}{i} \;=\; \sum_{k=1}^{\infty}\frac{1}{k}\Big(\frac{1}{e^{-2\pi ikz}-1} - \frac{1}{q^{-2\pi ik/z}-1}\Big)$$

Let

$$f(w) \;=\; \cot w\cdot\cot w/z$$

and let ν run over values $(n+\frac{1}{2})\pi$ for $0 \le n \in \mathbb{Z}$. Then $f(\nu w)/w$ has *simple* poles at $w = \pm\pi k/\nu$ and at $w = \pm\pi kz/\nu$, with respective residues

$$\frac{1}{\pi k}\cot\frac{\pi k}{z} \qquad\text{and}\qquad \frac{1}{\pi k}\cot\pi kz \qquad (\text{for } k = 1,2,3,\ldots)$$

and a *triple* pole at at $w = 0$ with residue $-\frac{1}{3}(z+z^{-1})$. Let γ be the path tracing counter-clockwise the outline of the parallelogram with vertices $1, z, -1, -z$. By residues,

$$\pi\frac{z+z^{-1}}{12} + \int_\gamma f(\nu w)\frac{dw}{8w} \;=\; \frac{i}{2}\sum_{k\ge 1}\frac{1}{k}(\cot\pi kz + \cot\pi k/z) \;=\; \sum_{k=1}^{\infty}\frac{1}{k}\Big(\frac{1}{q^{-k}-1} - \frac{1}{q^{-k/z}-1}\Big)$$

The parameters n or ν only appear in the contour integral on the left-hand side. To evaluate it, let as $n \to +\infty$. In this limit, $f(\nu w)$ is uniformly bounded on γ, and has limiting values on the sides (excluding the vertices, where there are discontinuities) $1, -1, 1, -1$, respectively. The limit of the contour integral is

$$\int_\gamma f(\nu w)\frac{dw}{8w} \;=\; \Big(\int_1^z - \int_z^{-1} + \int_{-1}^{-z} - \int_{-z}^{1}\Big)\frac{dw}{w} \;=\; 4\,\log\frac{z}{i}$$

This gives the functional equation. ///

---

2. *Weil's proof*

As in [Weil 1968], consider the Dirichlet series [1]

$$L(s) \;=\; \zeta(s)\cdot\zeta(s+1) \;=\; \sum_{m,n} \frac{1}{m}\frac{1}{(mn)^s} \;=\; \sum_{n=1}^\infty \Big(\sum_{d|n}\frac{1}{d}\Big)\frac{1}{n^s}$$

The *completed* version

$$\Lambda(s) \;=\; (2\pi)^{-s}\,\Gamma(s)\,L(s)$$

has functional equation inherited from $\zeta(s)$:

$$\Lambda(-s) \;=\; \Lambda(s)$$

Noting that $\zeta(2)/2\pi = \pi/12$,

$$\Lambda(s) \;=\; \frac{\pi/12}{s-1} - \frac{1}{2s^2} - \frac{\pi/12}{s+1} + (\text{holomorphic})$$

The power series in $q = e^{2\pi iz}$ with the same coefficients is

$$F(z) \;=\; \sum_{m,n}\frac{1}{m}q^{mn} \;=\; \sum_n \Big(\sum_m \frac{1}{m}\,(q^n)^m\Big) \;=\; -\sum_n \log(1-q^n)$$

Recall Dedekind's eta-function

$$\eta(z) \;=\; q^{1/24}\prod_{n\geq 1}(1-q^n)$$

Then

$$F(z) \;=\; -\Big(\frac{\log q}{24} + \sum_n \log(1-q^n)\Big) \;+\; \frac{\log q}{24} \;=\; -\log\eta \;+\; \frac{\pi iz}{12}$$

From the obvious Fourier-Mellin transform relation

$$\Lambda(s) \;=\; \int_0^\infty y^s\,F(iy)\,\frac{dy}{y} \qquad\qquad (\text{for } \mathrm{Re}(s) > 1)$$

Fourier-Mellin inversion gives

$$F(z) \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(z/i)^{-s}\,\Lambda(s)\,ds \qquad\qquad (\text{for } \sigma > 1)$$

Following Hecke and Weil, move the vertical line to $\mathrm{Re}(s) = -\sigma$, picking up residues at $1, 0, -1$:

$$F(z) \;=\; \int_{-\sigma-i\infty}^{-\sigma+i\infty}(z/i)^{-s}\,\Lambda(s)\,ds + \Big(\frac{\pi}{12}\cdot(z/i)^{-1} + \tfrac{1}{2}\log(z/i) - \frac{\pi}{12}\cdot(z/i)\Big)$$

$$= \int_{-\sigma-i\infty}^{-\sigma+i\infty}(z/i)^{-s}\,\Lambda(s)\,ds + \frac{\pi i}{12z} + \tfrac{1}{2}\log(z/i) - \frac{\pi z}{12i}$$

---

[1] Weil was well aware that $L(s)$ is essentially the Mellin transform of the *constant coefficient* in the Laurent expansion in s at $s = 1$ of the Eisenstein series $E_s = \sum \frac{y^s}{|cz|+d|^{2s}}$. The nature of that constant coefficient is part of the *Kronecker limit formula*.

The functional equation $\Lambda(-s) = \Lambda(s)$ allows conversion of the integral on $\mathrm{Re}(s) = -\sigma$ into

$$\frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} (z/i)^{-s}\,\Lambda(-s)\,ds \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Big(\frac{-1}{z/i}\Big)^{-s}\,\Lambda(s)\,ds \;=\; F(-1/z)$$

That is,

$$F(z) \;=\; F(-1/z) + \frac{\pi i}{12z} + \tfrac{1}{2}\log(z/i) - \frac{\pi z}{12i}$$

Using $F(z) = \pi iz/12 - \log\eta(z)$, this is

$$\frac{\pi iz}{12} - \log\eta(z) \;=\; \frac{\pi i(-1/z)}{12} - \log\eta(-1/z) + \frac{\pi i}{12z} + \tfrac{1}{2}\log(z/i) - \frac{\pi z}{12i}$$

which simplifies to

$$\log\eta(z) \;=\; \log\eta(-1/z) - \tfrac{1}{2}\log(z/i)$$

Exponentiating and taking the 24^{th} power:

$$\eta^{24}(z) \;=\; \eta^{24}(-1/z)\cdot(z/i)^{-12}$$

or

$$\eta^{24}(-1/z) \;=\; z^{12}\cdot\eta^{24}(z)$$

That is, η^{24} has the two functional equations

$$\eta^{24}(z+1) \;=\; \eta^{24}(z) \qquad\qquad \eta^{24}(-1/z) \;=\; z^{12}\cdot\eta^{24}(z)$$

and goes to 0 as $\mathrm{Im}(z) \to +\infty$. Since $SL_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

giving the maps $z \to -1/z$ and $z \to z+1$, evidently η is a *holomorphic cuspform* of weight 12, with leading Fourier coefficient 1. Thus, it is $\Delta(z)$, and we have the product expansion

$$\Delta(z) \;=\; \eta^{24}(z) \;=\; e^{2\pi iz}\prod_{n\ge 1}(1 - e^{2\pi inz})$$

[2.0.1] Remark: In fact, Weil's connection between a simple converse theorem and a product formula is anomalous.

[Hurwitz 1881] A. Hurwitz, *[thesis]*, Math. Ann. **18** (1881), 528-592.

[Siegel 1954] C.L. Siegel, *A simple proof of* $\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/i}$, Mathematika **1** (1954), p. 4 (Ges. Abh. Bd. III, Springer 1966, p. 188).

[Weil 1967] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149-156.

[Weil 1968] A. Weil, *Sur une formule classique*, J. Math. Soc. Japan **20** (1968), 400-402. (*Oeuvres Sci.* Vol. III, Springer, New York, 1979, 198-200.)
