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# Special values, Laurent coefficients, algebraic relations

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## 1. Special values $\zeta(2n)$ in a Laurent expansion

Via Liouville's theorem, by cancelling poles, and so on,

$$f(z) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} \qquad\qquad \text{(secretly } f(z)=\frac{\pi^2}{\sin^2 \pi z}\text{, but we don't use this)}$$

satisfies

$$f'^2 \;=\; 4f^2(f-\pi^2)$$

The Laurent coefficients of $f(z)$ at 0 have direct relations to the *special values* $\zeta(2),\zeta(4),\ldots$, producing algebraic relations among these values, as follows.

Let $g(z)=f(z)-\frac{1}{z^2}$, so $g(z)$ is holomorphic at $z=0$, and

$$g(z) \;=\; g(0)+\frac{g'(0)}{1!}z+\frac{g''(0)}{2!}z^2+\ldots$$

$$=\sum_{n\neq 0}\frac{1}{n^2}+\sum_{n\neq 0}\frac{-2}{1!\cdot n^3}z+\sum_{n\neq 0}\frac{(-2)(-3)}{2!\cdot n^4}z^2+\sum_{n\neq 0}\frac{(-2)(-3)(-4)}{3!\cdot n^5}z^3+\sum_{n\neq 0}\frac{(-2)(-3)(-4)(-5)}{4!\cdot n^6}z^4+\ldots$$

In the odd-degree sums the $\pm n$ terms cancel, giving

$$f(z) \;=\; \frac{1}{z^2}+2\zeta(2)+6\zeta(4)z^2+10\zeta(6)z^4+14\zeta(8)z^6+\ldots$$

and

$$f'(z) \;=\; \frac{-2}{z^3}+12\zeta(4)z+40\zeta(6)z^3+84\zeta(8)z^5+\ldots$$

The simplified relation $f'^2 = 4f^2(f-\pi^2)$ from above gives a recursion to determine $\zeta(2n)$ from $\zeta(2),\zeta(4),\ldots,\zeta(2n-2)$, for $2n\geq 6$, since all the Laurent coefficients of $0=f'^2-4f^2(f-\pi^2)$ vanish: namely, the first/lowest-degree term involving $\zeta(2n)$ is the $z^{2n-6}$ term

$$0 \;=\; 2\cdot\frac{-2}{z^3}\cdot(4n-2)(2n-2)\zeta(2n)\,z^{2n-3}-4\cdot 3\cdot\Big(\frac{1}{z^2}\Big)^2\cdot(4n-2)\zeta(2n)\,z^{2n-2}+\text{(previous)}$$

$$=\;-\Big(4(2n-2)+12\Big)(4n-2)\cdot\zeta(2n)+\text{(previous)} \;=\; -(8n+4)(4n-2)\cdot\zeta(2n)+\text{(previous)}$$

where *previous* is a polynomial involving $\zeta(2),\zeta(4),\ldots,\zeta(2n-2)$.

In fact, given that $\zeta(2)=\pi^2/6$ and $\zeta(4)=\pi^4/90$, this approach can prove

**[1.0.1] Claim:** $\zeta(2n)/\pi^{2n}$ is *rational*, for $2n=2,4,6,8,\ldots$.

***Proof:*** Rewrite the relation in terms of normalizations $\zeta(2m)/\pi^{2m}$. From the Laurent expansion,

$$\pi^{-2}f(z/\pi) \;=\; \frac{1}{z^2}+\frac{2\zeta(2)}{\pi^2}+\frac{6\zeta(4)}{\pi^4}z^2+\frac{10\zeta(6)}{\pi^6}z^4+\frac{14\zeta(8)z^6}{\pi^8}+\ldots$$

Replacing $f(z)$ by $F(z) = \pi^{-2}f(z/\pi)$ gives [1] $F'(z) = \pi^{-3}f'(z/\pi)$, and the relation $f'^2 = 4f^2(f-\pi^2)$ becomes

$$\pi^6 F'^2 \;=\; 4\pi^4 F^2(\pi^2 F - \pi^2)$$

giving a relation with *rational* coefficients:

$$F'^2 \;=\; 4F^2(F-1)$$

This relation gives a recursion with *rational* coefficients for the values $\zeta(2n)/\pi^{2n}$. Non-vanishing of the coefficient of $\zeta(2n)$ at its first appearance was checked above, so the recursion does not collapse. ///

**[1.0.2] Remark:** The above discussion clumsily mirrors more direct expression of special values of $\zeta$ in terms of *Bernoulli numbers*, better seen via Riemann's keyhole/Hankel contour expression for $\zeta(-n)$ and the *functional equation* for $\zeta(s)$. Nevertheless, it has some interest as a warm-up for the following example.

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## 2. *Special values* $L(2n,\chi)$ *in Laurent expansion of* $\wp(z)$

As the algebraic relation $f'^2 = 4f^2(f-\pi^2)$ for $f(z) = \sum_n 1/(z+n)^2$ gave relations among the Laurent coefficients of $f$ involving special values $\zeta(2n)$, the Weierstraß relation $\wp'^2 = 4\wp^3 - 60g_2\wp - 140g_3$ gives relations among the Laurent coefficients of $\wp(z)$. These Laurent coefficients are less elementary than the special values $\zeta(2n)$. For special lattices these are special values of *Hecke L-functions*, discussed below.

With fixed lattice $\Lambda$,

$$\wp(z) \;=\; \frac{1}{z^2} + \sum_{0\neq\lambda\in\Lambda}\Big(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\Big) \;=\; \frac{1}{z^2} + (\text{holomorphic at } z=0)$$

With $g(z) = \wp(z) - \frac{1}{z^2}$,

$$g(z) \;=\; g(0) + \frac{g'(0)}{1!}z + \frac{g''(0)}{2!}z^2 + \ldots$$

$$= \sum_{\lambda\neq 0}\Big(\frac{1}{\lambda^2} - \frac{1}{\lambda^2}\Big) + \sum_{\lambda\neq 0}\frac{-2}{1!\cdot\lambda^3}z + \sum_{\lambda\neq 0}\frac{(-2)(-3)}{2!\cdot\lambda^4}z^2 + \sum_{\lambda\neq 0}\frac{(-2)(-3)(-4)}{3!\cdot\lambda^5}z^3 + \sum_{\lambda\neq 0}\frac{(-2)(-3)(-4)(-5)}{4!\cdot\lambda^6}z^4 + \ldots$$

In the odd-degree sums the $\pm\lambda$ terms cancel, so

$$\wp(z) \;=\; \frac{1}{z^2} + \sum_{\lambda\neq 0}\frac{(-2)(-3)}{2!\cdot\lambda^4}z^2 + \sum_{\lambda\neq 0}\frac{(-2)(-3)(-4)(-5)}{4!\cdot\lambda^6}z^4 + \ldots \;=\; \frac{1}{z^2} + \sum_{n\geq 2}(2n-1)g_n z^{2n-2}$$

with $g_n = \sum_{0\neq\lambda\in\Lambda}\frac{1}{\lambda^{2n}}$. The Weierstraß relation gives a recursion for $g_4, g_5, \ldots$ in terms of $g_2, g_3$: the lowest-degree coefficient in which $g_n$ appears is that of $z^{2n-6}$, and this is

$$0 \;=\; \Big(2\cdot\frac{-2}{z^3}\cdot(2n-2)(2n-1)g_n z^{2n-3}\Big) - \Big(3\cdot\frac{1}{z^4}\cdot(2n-1)g_n z^{2n-2}\Big) + (\text{previous})$$

$$= -(2n-1)\Big(4(2n-2)+3\Big)g_n z^{2n-6} + (\text{previous}) \;=\; -(2n-1)(8n-5)g_n z^{2n-6} + (\text{previous})$$

The coefficient of $g_n$ is non-zero, so $g_n$ is a polynomial in $g_2, g_3, \ldots, g_{n-1}$ with rational coefficients, independent of the lattice.

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[1] Since $f(z) = \sum\frac{1}{(z+n)^2} = \frac{\pi^2}{\sin^2\pi z}$, in fact $F(z) = \frac{1}{\sin^2 z}$.

**[2.1] Some Hecke $L$-functions** The lattice $\Lambda = \mathbb{Z}\cdot i + \mathbb{Z}$ is the ring $\mathbb{Z}[i]$ of Gaussian integers. It is *Euclidean*, so is a principal ideal domain. The Galois norm is $N(a+bi) = a^2+b^2$, and *units* in $\mathbb{Z}[i]$ must have norm $\pm 1$, so the only units are $\pm 1, \pm i$.

The *Dedekind zeta function* for $\mathfrak{o} = \mathbb{Z}[i]$ is

$$\zeta_{\mathfrak{o}}(s) \;=\; \sum_{0\neq\alpha\in\mathfrak{o}/\mathfrak{o}^\times} \frac{1}{|\alpha|^{2s}}$$

The ring $\mathbb{Z}[i]$ has multiplicative [2] maps to the unit circle in $\mathbb{C}^\times$, namely

$$\chi \;:\; \alpha \;\longrightarrow\; \Big(\frac{\alpha}{|\alpha|}\Big)^n$$

For various reasons, we want $\chi$ to be invariant by units, that is, $\chi(\eta\cdot\alpha) = \chi(\alpha)$ for units $\eta \in \{\pm 1, \pm i\}$, entailing that $\chi$ be of the form $\chi_{4n}(\alpha) = (\alpha/|\alpha|)^{4n}$. With such $\chi$, the corresponding *unramified Hecke $L$-functions* for $\mathbb{Z}[i]$ are

$$L(s,\chi_{4n}) \;=\; \sum_{0\neq\alpha\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{\chi_{4n}(\alpha)}{|\alpha|^{2s}} \;=\; \sum_{0\neq\alpha\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{(\alpha/|\alpha|)^{4n}}{|\alpha|^{2s}}$$

Meanwhile, the functions $g_n = g_n(\mathbb{Z}[i])$ for this lattice are

$$g_n \;=\; \sum_{a,b\in\mathbb{Z}^2-(0,0)} \frac{1}{(a+bi)^{2n}} \;=\; 4\cdot \sum_{0\neq\alpha\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{\alpha^{2n}} \qquad\qquad \text{(vanishing unless } n\in 2\mathbb{Z})$$

Thus,

$$\frac{1}{4}g_{2n} \;=\; \sum_{0\neq\alpha\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{\alpha^{4n}} \;=\; \sum_{0\neq\alpha\in\mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{(\alpha/|\alpha|)^{-4n}}{|\alpha|^{2\cdot 2n}} \;=\; L(2n,\chi_{-4n})$$

This is a *special value* of $L(s,\chi_{-4n})$. In this example, $g_3 = 0$, so $g_4, g_6, \ldots$ are polynomials in $g_2$ with *rational* coefficients. That is, the special values $L(4,\chi_{-8}), L(6,\chi_{-12}), \ldots$ are polynomials in $L(2,\chi_{-4})$ with rational coefficients.

**[2.1.1] Remark:** A similar discussion applies to lattices $\Lambda = \mathbb{Z}\cdot z + \mathbb{Z}$ where $\mathbb{Z}[z]$ is the ring of algebraic integers in a quadratic extension $\mathbb{Q}(z)$ of $\mathbb{Q}$.

**[2.1.2] Remark:** The idea is that, just as the special values $\zeta(2n)$ are rational except for appearance of the single transcendental $\pi$, the lists of special values $L(2n,\chi)$ need fewer transcendentals than expected.

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[2] As usual, a map $\chi : \mathbb{Z}[i] \to \mathbb{C}^\times$ is *multiplicative* when $\chi(\alpha\cdot\beta) = \chi(\alpha)\cdot\chi(\beta)$ for all $\alpha,\beta\in\mathbb{Z}[i]$.

## Bibliography

The examples above are a very tiny and idiosyncratic sample of ideas about *special values* of $L$-functions. The literature on special values is huge and still growing. For perspective, only through the mid-1970s:

The method of [Riemann 1859] suffices for $\zeta(s)$ itself, and for the Dirichlet $L$-functions introduced in 1837. Continuing [Blumenthal 1903/4]'s discussion of Hilbert-Blumenthal modular forms, [Hecke 1922/24] conjectured special-values results for Dedekind zeta functions. [Siegel 1937] proved this conjecture, and [Klingen 1961/2] gave a simpler proof using Hilbert-Blumenthal modular forms. [Damerell 1970/71] studied special values of $L$-functions in a special class including those we discussed above via the Weierstraß equation. Damerell's argument was simplified in [Weil 1976]. [Shimura 1975] and [Shimura 1976] greatly extended investigations of special-values. The results known at the time were put into a very broad conjectural framework by [Deligne 1977/9].

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