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Failure of the minimum principle in Banach spaces

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/11b_failure_minimum_principle.pdf]

First, some related points about the discrepancies between C^o, C^1, and L^2:

• It is not hard to find sequences in $C^o[a,b]$ converging to 0 in L^2-metric but not convergent in $C^o[a,b]$. This shows that the L^2 topology is *strictly weaker* than the sup-norm C^o topology.

• Existence of a *dense* subset of $C^o[a,b]$ of *nowhere differentiable* functions follows from the Baire category theorem: Let U_n be the subset of $C^o[a,b]$ consisting of f such that for *all* $t \in [a,b]$

$$|f(s) - f(t)| \;>\; n \cdot |s - t|$$

for *some* $s \in [a,b] \cap [t - \frac{1}{n}, t + \frac{1}{n}]$. We show U_n is *open* and *dense*, so by the Baire category theorem, $\bigcap_n U_n$ is *dense* and consists of *nowhere differentiable* continuous functions.

Two illustrations of the non-pathological failure of a minimum principle in *Banach* spaces (as opposed to Hilbert spaces):

• Let E be the set of continuous functions $f \in C^o[0,2]$ with

$$\int_0^1 f\,dx - \int_1^2 f\,dx = 1$$

The set E is *closed* and *convex*, but has *no* element of minimal norm (in contrast to the Hilbert space situation).

• Let E be the functions f in $L^1[0,1]$ such that $\int_0^1 f\,dx = 1$. Then E is a closed convex subset with *infinitely many* elements of minimal norm (in contrast to the Hilbert space situation).
