Complex numbers

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

Complex analysis is one of the most natural and productive continuations of basic calculus, not addressing pathologies and pitfalls, but, instead, showing that natural functions have even better properties than imagined. The first 150 years of calculus addressed such nice functions, discovering the remarkable usefulness of calculus-as-complex-analysis implicitly, long before anyone worried about the subtleties and troubles highlighted in the 19th century. Euler, Lagrange, and the most effective of their contemporaries, inadvertently thought in terms we can now reinterpret as part of complex analysis, by often considering function to mean expressible as (convergent?) power series.

Complex numbers such as $\sqrt{-1}$ entered not by a perceived need to solve quadratic equations like $x^2 + 1 = 0$ lacking real-number roots, since such equations could simply be declared to have no roots, and forget about the whole difficulty. For that matter, negative numbers were considered merely convenient fictions by many mathematicians until just a few hundred years ago. Rather, an unavoidable awkwardness forced consideration of complex numbers, namely, the expression in radicals for roots of cubic equations $x^3 + bx + c = 0$ is something like

$$
\text{roots} = \frac{\lambda + \lambda'}{3}, \quad \frac{\omega \lambda + \omega^2 \lambda'}{3}, \quad \frac{\omega^2 \lambda + \omega \lambda'}{3}
$$

where

$$
\lambda = \sqrt[3]{3bc - \frac{27}{2}c + (3\omega - \frac{1}{2})\sqrt[3]{\Delta}}, \quad \lambda' = \sqrt[3]{3bc - \frac{27}{2}c + (3\omega^2 - \frac{1}{2})\sqrt[3]{\Delta}}
$$

$$
\omega = -\frac{1 + \sqrt{-3}}{2} = \text{cube root of unity}
$$

and, letting $\alpha, \beta, \gamma$ be the roots of the equation,

$$
\Delta = \text{discriminant of cubic} = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = -27c^2 - 4b^3
$$

The awkwardness is that the cubic can have three real roots, thus inarguably legitimate, but the expression for them requires complex numbers.

Although Abel and Jacobi’s work on elliptic integrals and elliptic functions (discussed later) necessarily referred to complex numbers, Cauchy’s work on the basic calculus of complex analysis from the 1820s on showed its unexpected power, and made it indispensable to physicists and electrical engineers. Also, Cauchy argued for the genuine-ness of complex numbers by promoting the now-familiar pictorial representation as two-dimensional numbers corresponding to points in the plane, just as real numbers are construed as corresponding to points on the line. Indeed, Hamilton’s later creation of quaternions was substantially motivated by an impulse to identify higher-dimensional numbers. Cauchy’s basic complex analysis, as refined by Goursat, will be the first substantive body of results discussed, after some set-up.

Riemann’s 1858/9 paper on the zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ showed that complex analysis was definitely not just a different fashion for doing multi-variable calculus, as one might insist on viewing Cauchy’s work. Specifically, beginning with Euler’s discovery of the factorization

$$
\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \quad \text{(for Re}(s) > 1)
$$

(proven by expanding each $1/(1-p^{-s})$ as a geometric series, multiplying these all together, and using unique factorization). Euler had already observed that this factorization gives a proof of the infinitude of primes more quantitative than the Euclidean proof, namely, that the sum expression blows up like $1/(s - 1)$ as
$s \to 1^+$, so the product must do likewise. Riemann sketched an argument that (in von Mangoldt’s modified form) the sum of $\log p$ for prime powers $p^m$ below a bound $T$ is exactly expressible in terms of the complex zeros $\rho$ of $\zeta(s)$:

$$\sum_{\text{prime power } p^m < T} \log p = T - c - \sum_{\text{complex } \rho : \zeta(\rho) = 0} \frac{T^n}{\rho} + \sum_{n \geq 1} \frac{T^{-2n}}{2n}$$

with a concrete but irrelevant constant $c$. Indeed, the analytic continuation of $\zeta(s)$ to the whole complex plane $\mathbb{C}$ vanishes at the negative even integers $-2n$, and these are the trivial zeros of $\zeta(s)$.

This exact equality, known as Riemann’s Explicit Formula, is of a completely different nature than the merely asymptotic assertion in the Prime Number Theorem, with or without error estimates, that

$$\frac{\text{number of primes } < T}{T / \log T} \to 1 \quad (\text{as } T \to +\infty)$$

In fact, the latter far weaker result itself, although conjecture prior to 1800, was only proven 40 years after Riemann’s paper, by Hadamard and de la Vallée Poussin, independently.

Indeed, filling out and legitimizing the arguments sketched in Riemann’s paper was a central motivation in much of the development of complex analysis through the late 19th century and into the 20th.