05. The Gamma function $\Gamma(s)$ [draft]

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1. Euler’s integral for $\Gamma(s)$

The Gamma function $\Gamma(s)$ can be defined by Euler’s integral

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t} \quad \text{(absolute convergence for } \text{Re}(s) > 0)$$

Integration by parts proves the functional equation

$$\Gamma(s + 1) = s \cdot \Gamma(s) \quad \text{ (for } \text{Re}(s) > 0)$$

For $0 < s \in \mathbb{Z}$, the functional equation and induction show the connection to factorials:

$$\Gamma(n) = (n - 1)! \quad \text{ (for } n = 1, 2, \ldots)$$

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2. Holomorphy of integrals

It is not surprising that $\Gamma(s)$ is holomorphic in the region of absolute convergence $\text{Re}(s) > 0$. This can be proven by checking complex differentiability of truncated integrals, and invoking the holomorphy of uniform-on-compact limits of holomorphic functions. Alternatively, but essentially equivalently in terms of fundamental invocation of Cauchy’s theorem and corollaries, holomorphy can be proven via Morera’s theorem, invoking Fubini-Tonelli to justify interchange of integrals. Both approaches are typical for proving holomorphy of integrals with a parameter, when the integrands are holomorphic functions of the parameter. In this section we recall some broadly applicable ideas.

[2.1] Claim: Let $F(t, z)$ be a function of $t \in [a, b] \subset \mathbb{R}$ and $z \in \Omega \subset \mathbb{C}$ with non-empty open $\Omega$, continuous as a function of the two variables, and holomorphic in $z$ for each fixed $t$. Then

$$f(z) = \int_a^b F(t, z) \, dt$$

is holomorphic for $z \in \Omega$. Further, the complex derivative is

$$f'(z) = \int_a^b \frac{\partial F}{\partial z}(t, z) \, dt$$
where \( \frac{\partial F}{\partial z} \) is the complex derivative in the second argument of \( F \). That is, the operator of complex differentiation passes inside the integral.

[2.2] **Remark:** Without compactness or similar hypothesis on the behavior in the integration variable, the conclusion can easily fail, and in non-pathological ways, for example,

\[
f(z) = \int_{-\infty}^{\infty} \frac{e^{itz}}{1 + t^2} dt
\]

is *not* holomorphic in \( z \). The integral does not converge at all for \( z \not\in \mathbb{R} \).

**Proof:** First, we claim that \( F_z \) and \( F_{zz} \), the first and second complex derivatives of \( F \) in its second argument, are continuous as functions of their two arguments. From Cauchy's integral formulas, for each fixed \( t \in [a, b] \), for any simple closed path \( \gamma \) around \( z_o \), inside \( \Omega \), for any \( z \) inside \( \gamma \),

\[
F(t, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(t, w) \, dw}{w - z} \quad \text{and} \quad F_z(t, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{F(t, w) \, dw}{(w - z)^2}
\]

and similarly for \( F_{zz} \). Given \( z_o \in \Omega \), let \( B_{2r}, B_r \) be open balls of radius \( 2r, r \) centered at \( z_o \) and so that the closure of \( B_{2r} \) fits inside \( \Omega \). We may as well let \( \gamma \) be the boundary of \( B_{2r} \), traversed in a positive direction. Let \( C_r \) be the closure of \( B_r \). The continuity of \( F \) on the compact set \([a, b] \times C_r\) implies *uniform* continuity on that set, and on \([a, b] \times \gamma\).

Using that joint continuity, given \( \varepsilon > 0 \), take \( \delta > 0 \) such that \( |z - z_o| < \delta \) implies \( |(w - z)^{-2} - (w - z_o)^{-2}| < \varepsilon \) for all \( w \in \gamma \). Let \( M \) be the maximum of the continuous function \( F(t, w) \) on the compact \([a, b] \times C_r\). By the trivial estimate on the Cauchy formula integral,

\[
\left| \int_{\gamma} \frac{F(t, w) \, dw}{(w - z)^2} - \int_{\gamma} \frac{F(t, w) \, dw}{(w - z_o)^2} \right| \leq 2\pi 2r \cdot M \cdot \max_{w \in \gamma} \left| \frac{1}{(w - z)^2} - \frac{1}{(w - z_o)^2} \right| \leq 2\pi 2r \cdot M \cdot \varepsilon
\]

This gives the continuity of \( F_z(t, z) \). A nearly identical argument gives that of \( F_{zz}(t, z) \).

By the complex differentiability in \( z \), for fixed \( z_o \), for every \( t \in [a, b] \) and \( z \in C_r \),

\[
F(t, z) = F(t, z_o) + (z - z_o)F_z(t, z_o) + R(t, z)
\]

where the remainder \( R(t, z) \) satisfies a uniform estimate of the form

\[
|R(t, z)| \leq B \cdot |z - z_o|^2 \quad \text{for all} \ (t, z) \in [a, b] \times C_r
\]

Thus,

\[
\left| \int_{[a, b]} F(t, z) \, dt - \int_{[a, b]} F(t, z_o) \, dt \right| \leq \int_{[a, b]} |F(t, z) - F(t, z_o)| \, dt \leq \int_{[a, b]} B \cdot |z - z_o|^2 \, dt = |b - a| \cdot B \cdot |z - z_o|^2
\]

Thus,

\[
\left| \int_{[a, b]} \frac{F(t, z) - F(t, z_o)}{z - z_o} - F_z(t, z_o) \, dt \right| \leq |b - a| \cdot B \cdot |z - z_o| \rightarrow 0 \quad \text{as} \ z \rightarrow z_o
\]

This proves the complex differentiability of the integral in \( t \), and identifies the derivative as the corresponding integral of \( F_z \). That is, the complex differentiation in \( z \) passes inside the integral, as hoped.

As an example of limits of compact integrals that are still holomorphic:
[2.3] Claim: Let $F(t, z)$ be continuous in $t \in (0, \infty)$ and complex differentiable in $z$ in non-empty open $\Omega$. Assume that

$$f(z) = \int_0^\infty F(t, z) \, dt$$

is absolutely convergent for all $z \in \Omega$. Assume that, for every compact $K \subset \Omega$,

$$\lim_{a \to 0^+, b \to +\infty} \int_a^b F(t, z) \, dt = \int_0^\infty F(t, z) \, dt \quad \text{(uniformly for } z \in K)$$

That is, given compact $K \subset \Omega$, given $\varepsilon > 0$, there exist $a_0, b_0$ such that, for all $z \in K$, and for all $0 < a, a' \leq a_0$ and for all $b, b' \geq b_0$,

$$\left| \int_a^b F(t, z) \, dt - \int_{a'}^{b'} F(t, z) \, dt \right| < \varepsilon$$

Then $\int_0^\infty F(t, z) \, dt$ is holomorphic in $z \in \Omega$, and its complex derivative is $\int_0^\infty F_z(t, z) \, dt$.

Proof: The previous claim shows that all the truncated integrals $f_{a,b}(z) = \int_a^b F(t, z) \, dt$ are holomorphic. The hypothesis is exactly that the functions $f_{a,b}$ converge pointwise, uniformly on compacts, to the infinite integral. A uniform-on-compacts pointwise limit of holomorphic functions is holomorphic. ///

3. Holomorphy of $\Gamma(s)$ in $\text{Re}(s) > 0$

The general claims of the previous section give

[3.1] Claim: The integral $\int_0^\infty e^{-t} t^s \frac{dt}{t}$ is a holomorphic function of complex $s$ for $\text{Re}(s) > 0$.

Proof: The cases that $0 < \text{Re}(s) \leq 1$ and $1 \leq \text{Re}(s)$ are somewhat different, due to the corresponding behaviors of $t^s$ near 0 and near $+\infty$.

For $\text{Re}(s) \geq 1$, with the logarithm that is real-valued on $(0, +\infty)$, for $0 < b < b'$,

$$|t^{s-1}| = |e^{(s-1) \log t}| = e^{\text{Re}((s-1) \log t)} = e^{(\text{Re}(s)-1) \log t} = t^{\text{Re}(s)-1}$$

Then

$$\left| \int_b^{b'} e^{-t} t^s \frac{dt}{t} \right| \leq \int_b^{b'} e^{-t} t\text{Re}(s) \frac{dt}{t} \leq \int_0^\infty e^{-t} t\text{Re}(s) \frac{dt}{t}$$

$$\leq \int_b^\infty e^{-t/2} e^{-t/2} t\text{Re}(s) \frac{dt}{t} = \int_b^\infty e^{-t/2} dt \times \sup_{t \geq b} e^{-t/2} t\text{Re}(s)-1 = b^{-t/2} \times \sup_{t \geq b} e^{-t/2} t\text{Re}(s)$$

Given compact $K$, there is $\sigma_1$ such that $s \in K$ implies $\text{Re}(s) \leq \sigma_1$. The sup is finite, so we have exponential decay in $b$, giving the uniform estimate

$$\left| \int_0^b e^{-t} t^s \frac{dt}{t} \right| \leq b^{-t/2} \times \sup_{t \geq b} e^{-t/2} t^{\sigma_1}$$

for $s \in K$. For $0 < \text{Re}(s) \leq 1$, the convergence of $\int_0^1 e^{-t} t^s \frac{dt}{t}$ implies that

$$\lim_{a,a' \to 0^+} \int_a^{a'} e^{-t} t^s \frac{dt}{t} \to 0$$

Then similar estimates give the uniform-on-compacts convergence. ///
4. Meromorphic continuation of $\Gamma(s)$ to $\mathbb{C}$

From the functional equation, we get a meromorphic continuation of $\Gamma(s)$ to the entire complex plane, except for poles at non-positive integers $-n$. The poles are simple, with residue $(-1)^n/n!$ at $-n$.

5. \[ \int_0^\infty e^{-tz} t^s \frac{dt}{t} = z^{-s} \Gamma(s) \]

The identity
\[ \int_0^\infty t^s e^{-ty} \frac{dt}{t} = \frac{\Gamma(s)}{y^s} \quad \text{(for } y > 0 \text{ and } \Re(s) > 0) \]
for $y > 0$ first follows for $\Re(s) > 0$ by replacing $t$ by $t/y$ in the integral. Then
\[ \int_0^\infty t^s e^{-tz} \frac{dt}{t} = \frac{\Gamma(s)}{z^s} \quad \text{(for } \Re(z) > 0 \text{ and } \Re(s) > 0) \]
by complex analysis, since both sides are holomorphic in $s$ and agree on the positive reals.

The latter identity allows non-obvious evaluation of a Fourier transform. Namely, let
\[ f(x) = \begin{cases} x^\alpha \cdot e^{-x} & \text{(for } x > 0) \\ 0 & \text{(for } x < 0) \end{cases} \]
For $\Re(\alpha) > -1$ this function is locally integrable at $0$, and in any case is of rapid decay at infinity. We can compute its Fourier transform:
\[ \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) \, dx = \int_0^\infty e^{-2\pi i \xi x} x^{\alpha+1} e^{-x} \frac{dx}{x} = \int_0^\infty x^{\alpha+1} e^{-x(1+2\pi i \xi)} \frac{dx}{x} = \frac{\Gamma(\alpha+1)}{(1+2\pi i \xi)^{\alpha+1}} \]
Further, Fourier inversion gives the non-obvious
\[ \int_{\mathbb{R}} e^{2\pi i \xi x} \frac{1}{(1+2\pi i \xi)^{\alpha+1}} \, d\xi = \frac{1}{\Gamma(\alpha+1)} \begin{cases} x^\alpha \cdot e^{-x} & \text{(for } x > 0) \\ 0 & \text{(for } x < 0) \end{cases} \]
For $\alpha \in \mathbb{Z}$, the same conclusion can be reached by evaluation by residues.

6. Euler’s Beta integral in terms of $\Gamma$

[6.1] Claim: Euler’s beta integral
\[ B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} \, dx \]
is expressible in terms of $\Gamma$ as
\[ B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} \, dx = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \]

Proof: Replacing $x$ by $\frac{t}{t+1} = 1 - \frac{1}{t+1}$ in the integral gives
\[ \int_0^1 x^{a-1} (1-x)^{b-1} \, dx = \int_0^\infty \left( \frac{t}{t+1} \right)^{a-1} \left( 1 - \frac{t}{t+1} \right)^{b-1} \frac{dt}{(t+1)^2} = \int_0^\infty t^a \left( \frac{1}{t+1} \right)^{a+b} \frac{dt}{t} \]
Use the gamma identity in the form

\[
\left( \frac{1}{t+1} \right)^s = \frac{1}{\Gamma(s)} \int_0^\infty e^{-u(t+1)} u^s \frac{du}{u}
\]

to rewrite the beta integral further as

\[
\frac{1}{\Gamma(a+b)} \int_0^\infty \int_0^\infty u^{a+b} t^a e^{-u(t+1)} \frac{du}{u} \frac{dt}{t} = \frac{1}{\Gamma(a+b)} \int_0^\infty \int_0^\infty u^a t^a e^{-t} \frac{dt}{t} \frac{du}{u} = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}
\]
as claimed.

\[6.2\] Remark: If we add another similar factor to the Beta integral, we have Euler’s integral representation for hypergeometric functions, namely,

\[
F(\alpha, \beta, \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} (1-xz)^{-\alpha} \, dx
\]

This \( F \) is the \( {}_2F_1 \) hypergeometric function, whose series definition is

\[
F(\alpha, \beta, \gamma; z) = 1 + \frac{abz}{c} + \frac{a(a+1)b(b+1)z^2}{2c(c+1)} + \cdots = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}
\]

The notation \((a)_n\) is the Pockhammer symbol.

7. \( \Gamma(s) \cdot \Gamma(1-s) = \pi / \sin \pi s \)

To prove this, take \( 0 < \text{Re}(s) < 1 \) for convergence of both integrals, and compute

\[
\Gamma(s) \cdot \Gamma(1-s) = \int_0^\infty \int_0^\infty u^s e^{-u} \cdot v^{1-s} e^{-v} \frac{du}{u} \frac{dv}{v} = \int_0^\infty \int_0^\infty u e^{-u(1+v)} v^{1-s} \frac{du}{u} \frac{dv}{v}
\]

by replacing \( v \) by \( uv \). Replacing \( u \) by \( u/(1+v) \) (another instance of the basic gamma identity) and noting that \( \Gamma(1) = 1 \) gives

\[
\int_0^\infty \frac{v^{-s}}{1+v} \, dv
\]

Replace the path from 0 to \( \infty \) by the Hankel contour \( H_\varepsilon \) described as follows. Far to the right on the real line, start with the branch of \( v^{-s} \) given by \((e^{2\pi i}v)^{-s} = e^{-2\pi isv^{-s}}\), integrate from \( +\infty \) to \( \varepsilon > 0 \) along the real axis, clockwise around a circle of radius \( \varepsilon \) at 0, then back out to \( +\infty \), now with the standard branch of \( v^{-s} \). For \( \text{Re}(-s) > -1 \) the integral around the little circle goes to 0 as \( \varepsilon \to 0 \). Thus,

\[
\int_0^\infty \frac{v^{-s}}{1+v} \, dv = \lim_{\varepsilon \to 0} \frac{1}{1-e^{-2\pi i s}} \int_{H_\varepsilon} v^{-s} \, dv
\]

The integral of this integrand over a large circle goes to 0 as the radius goes to \( +\infty \), for \( \text{Re}(-s) < 0 \). Thus, this integral is equal to the limit as \( R \to +\infty \) and \( \varepsilon \to 0 \) of the integral

from \( R \) to \( \varepsilon \)
from \( \varepsilon \) clockwise back to \( \varepsilon \)
from \( \varepsilon \) to \( R \)
from \( R \) counterclockwise to \( R \)

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This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $v = -1 = e^{\pi i}$. Thus,

$$\Gamma(s) \cdot \Gamma(1 - s) = \int_0^\infty \frac{v^{-s}}{1 + v} \, dv = \frac{2\pi i}{1 - e^{-2\pi is}} \cdot (e^{\pi i})^{-s} = \frac{2\pi i}{e^{\pi is} - e^{-\pi is}} = \frac{\pi}{\sin \pi s}$$

as claimed. ///

[7.1] Corollary: $\Gamma(s) \neq 0$ for all $s \in \mathbb{C}$. ///

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8. **Duplication**: $\Gamma(s) \cdot \Gamma(s + \frac{1}{2}) = 2^{1-2s} \cdot \sqrt{\pi} \cdot \Gamma(2s)$

To prove this, from the Eulerian integral definition,

$$\Gamma(s) \cdot \Gamma(s + \frac{1}{2}) = \int_0^\infty e^{-t} t^{s} \, dt \cdot \int_0^\infty e^{-u} u^{s+\frac{1}{2}} \, du$$

Replacing $t$ by $t/u$

$$\int_0^\infty \int_0^\infty e^{-\left(\frac{t}{u}+u\right)} t^{s} u^{\frac{1}{2}} \, du \, dt$$

In the Fourier transform identity

$$e^{-\pi \xi^2} = \int_{\mathbb{R}} e^{-2\pi i x \xi} e^{-\pi x^2} \, dx$$

let $\xi = \sqrt{t}/\sqrt{u}$ and replace $x$ by $x/\sqrt{\pi}$:

$$e^{-\frac{t}{u}} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2\pi i x \frac{\sqrt{t}}{\sqrt{u}}} e^{-x^2} \, dx$$

and replace $t$ by $t/\pi$ to obtain

$$e^{-\frac{t}{u}} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2\pi i x \frac{\sqrt{t}}{\sqrt{u}}} e^{-t} dx$$

Substituting the Fourier transform expression in place of $e^{-\frac{t}{u}}$ gives

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x \frac{\sqrt{t}}{\sqrt{u}}} e^{-x^2} e^{-u} t^{s} u^{\frac{1}{2}} \, dx \, du \, dt$$

Replace $x$ by $x\sqrt{u}$, and then $u$ by $u/(x^2 + 1)$:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x \sqrt{t}} e^{-u(x^2 + 1)} t^{s} u \, du \, dx \, dt = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x \sqrt{t}} \frac{1}{x^2 + 1} e^{-u} t^{s} u \, du \, dx \, dt$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1) \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x \sqrt{t}} \frac{1}{x^2 + 1} t^{s} dx \, dt = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i x \sqrt{t}} \frac{1}{x^2 + 1} t^{s} dx \, dt$$

The inner integral over $x$ can be evaluated by residues: it captures the negative of the residue of $x \to e^{-2\pi i x \sqrt{t}}/(x^2 + 1)$ in the lower half-plane, giving

$$\int_{\mathbb{R}} e^{-2\pi i x \sqrt{t}} \frac{1}{x^2 + 1} dx = -2\pi i \cdot e^{-2\pi i(-i)\sqrt{t}} \cdot \frac{1}{(-i) - i} = \pi e^{-2\sqrt{t}}$$

Summarizing, and then replacing $t$ by $t^2$ and $t$ by $t/2$:

$$\Gamma(s) \cdot \Gamma(s + \frac{1}{2}) = \sqrt{\pi} \int_0^\infty e^{-\sqrt{t} t^{s}} \frac{dt}{t} = 2\sqrt{\pi} \int_0^\infty e^{-2t} t^{2s} \frac{dt}{t} = 2^{1-2s} \sqrt{\pi} \int_0^\infty e^{-t} t^{2s} \frac{dt}{t} = \sqrt{\pi} 2^{1-2s} \Gamma(2s)$$

as claimed. ///