07. Compactification: Riemann sphere, projective space

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1. The Riemann sphere

One traditional one-point compactification of \( \mathbb{C} \) can be picturesquely extrinsically described via the stereographic projection map from the unit sphere \( S^2 \subset \mathbb{R}^3 \), with the point \((x,y,z) = (0,0,1)\) removed, to the \(x,y\)-plane. The same device applies to \( \mathbb{R}^n \), as follows.

The inverse stereographic projection map from \( \mathbb{R}^n \) to the unit sphere \( S^n \subset \mathbb{R}^{n+1} \) sends a point \( x = (x_1,\ldots,x_n) \in \mathbb{R}^n \) to the intersection point of the unit sphere \( S^n \subset \mathbb{R}^{n+1} \) with the line segment connecting \((x,0) = (x_1,\ldots,x_n,0)\) to the point \( p = (0,\ldots,0,1) \). Formulaically, this is

\[
\sigma : x \longrightarrow \left( \frac{2x_1}{|x|^2 + 1}, \ldots, \frac{2x_n}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1} \right) \quad \text{(for } x = (x_1,\ldots,x_n) \in \mathbb{R}^n)\]

where \(|x| = \sqrt{x_1^2 + \ldots + x_n^2}\) as usual. The inverse map is

\[
\sigma^{-1} : (y,z) = (y_1,\ldots,y_n,z) \longrightarrow \frac{y}{1-z} = \left( \frac{y_1}{1-z}, \ldots, \frac{y_n}{1-z} \right)
\]

and this certifies that \( \sigma \) is a smooth homeomorphism of \( \mathbb{R}^n \) with \( S^n - \{p\} \). Certainly \( S^n \) is compact.

Thus, the corresponding extrinsic one-point compactification of \( \mathbb{R}^n \) adjoins a point named \( \infty \), and declares the neighborhoods of \( \infty \) in \( \mathbb{R}^n \cup \{\infty\} \) to be the inverse images \( \sigma^{-1}(U - \{p\}) \) of punctured neighborhoods \( U - \{p\} \) of \( p \in S^n \).

A local basis at \( \infty \) consists of sets

\[
\{ \infty \} \cup \{ x \in \mathbb{R}^n : |x| > r \} \quad \text{(for } r \geq 0)\]

[1.1] Remark: A notable failing of this extrinsic stereographic compactification of \( \mathbb{C} \approx \mathbb{R}^2 \) is that it does not help describe the complex structure at the new point \( \infty \), so that we have no immediate sense of functions’ holomorphy at infinity or meromorphy at infinity.

[1] In general, a one-point compactification of a Hausdorff topological space \( X \) can be described intrinsically, without imbedding in a larger space and without comparison to a pre-existing compact space: let \( \tilde{X} = X \cup \{\infty\} \), and neighborhoods of \( \infty \) are all sets in \( \tilde{X} \) of the form \( \tilde{X} - K \) where \( K \) is a compact subset of \( X \), noting that Hausdorffness implies that compact sets are closed.
2. The complex projective line $\mathbb{CP}^1$

For purposes of complex analysis, a better description of a one-point compactification of $\mathbb{C}$ is an instance of the complex projective space $\mathbb{CP}^n$, a compact space containing $\mathbb{C}^n$, described as follows. Let $\sim$ be the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ by $x \sim y$ when $x = \alpha \cdot y$ for some $\alpha \in \mathbb{C}^\times$. Thus, $x \sim y$ means that $x$ and $y$ lie on the same complex line inside $\mathbb{C}^{n+1}$. The complex projective $n$-space $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by this equivalence relation:

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right)/\sim \approx \{\text{complex lines in } \mathbb{C}^{n+1}\}$$

Every equivalence class in $\mathbb{CP}^n$ has a representative in the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, and the further map to $\mathbb{CP}^n$ is continuous, so $\mathbb{CP}^n$ is compact.

There is the inclusion $\mathbb{C}^n \to \mathbb{CP}^n$ by $z = (z_1, \ldots, z_n) \mapsto \text{equivalence class of } (z_1, \ldots, z_n, 1) = \mathbb{C}^\times \cdot (z_1, \ldots, z_n, 1)$

The image of $\mathbb{C}^n$ in $\mathbb{CP}^n$ misses exactly

$$\{(z_1, \ldots, z_n, 0)\}/\sim \approx \mathbb{CP}^{n-1}$$

For $n = 1$, this is the single point

$$\infty = \{(z_1, 0)\}/\sim \approx \mathbb{CP}^0 \approx \{\text{pt}\}$$

so $\mathbb{CP}^1$ is a one-point compactification of $\mathbb{C}$. Otherwise, $\mathbb{CP}^n$ is strictly bigger than a one-point compactification.

Homogeneous coordinates on $\mathbb{CP}^n$ are the coordinates on $\mathbb{C}^{n+1}$ for representatives of the quotient. Thus, for $\mathbb{C} \subset \mathbb{CP}^1$, the homogeneous coordinates for the image of $z$ are $\left(\begin{array}{c} z \\ 1 \end{array}\right)$, for example. Going in the other direction, given homogeneous coordinates $\left(\begin{array}{c} u \\ v \end{array}\right)$, for $v \neq 0$, this represents the same equivalence class as does $\left(\begin{array}{c} u/v \\ 1 \end{array}\right)$, which is the image of the point $u/v \in \mathbb{C}$. If $v = 0$, then necessarily $u \neq 0$, and $\left(\begin{array}{c} u \\ 0 \end{array}\right) \sim \left(\begin{array}{c} 1 \\ 0 \end{array}\right)$ is $\infty$, the point at infinity.

[2.1] Remark: This procedure gives $\mathbb{CP}^n$ a natural complex structure for all $n$, as illustrated in the $n = 1$ case just below, in contrast to the stereographic one-point compactification. However, even for $n = 1$, the meaning of complex structure will be considered at length only somewhat later, in discussion of (complex-) one-dimensional complex manifolds, also known as Riemann surfaces.

3. Functions’ behavior at infinity

At least as a preliminary version, for a function $f$ holomorphic in a region $|z| > r$

$$\begin{align*}
is \text{ holomorphic at } \infty & \iff z \to f(1/z) \text{ is holomorphic at } 0 \\
is \text{ meromorphic at } \infty & \iff z \to f(1/z) \text{ is meromorphic at } 0 \\
\text{has an essential singularity at } \infty & \iff z \to f(1/z) \text{ has an essential singularity at } 0
\end{align*}$$
This is consistent with the one-point compactification’s topology, declaring the neighborhoods of $\infty$ to be complements of compact subsets of $\mathbb{C}$ (with $\infty$ added), so mapping $z \to 1/z$ maps punctured neighborhoods of 0 to punctured neighborhoods of $\infty$, and vice-versa.

For example,

\[
\begin{align*}
\text{behavior of } z & \to z^2 \text{ at } \infty \iff \text{behavior of } z & \to \frac{1}{z^2} \text{ at } 0 & \text{(meromorphic)} \\
\text{behavior of } z & \to \frac{1}{z^2} \text{ at } \infty \iff \text{behavior of } z & \to z^2 \text{ at } 0 & \text{(holomorphic)} \\
\text{behavior of } z & \to \frac{z-1}{z+1} \text{ at } \infty \iff \text{behavior of } z & \to \frac{\frac{z-1}{z+1}}{\frac{1}{z+1}} = \frac{1-z}{1+z} \text{ at } 0 & \text{(holomorphic)} \\
\text{behavior of } z & \to e^z \text{ at } \infty \iff \text{behavior of } z & \to e^{1/z} = \ldots + \frac{1}{z^2} + \frac{1}{z} + 1 \text{ at } 0 & \text{(ess sing)}
\end{align*}
\]

[3.1] Claim: The functions holomorphic on the whole $\mathbb{CP}^1$ are just constants. The functions $f$ meromorphic on the whole $\mathbb{CP}^1$ are exactly rational functions $f(z) = \frac{P(z)}{Q(z)}$, with polynomials $P, Q$ and $Q$ not identically 0.

Proof: For $f$ to be holomorphic at $\infty$ means that $z \to f(1/z)$ is holomorphic near 0. In particular, it is bounded on some neighborhood $|z| < \varepsilon$ of 0. Then $z \to f(z)$ is bounded on $|z| > 1/\varepsilon$. Certainly $z \to f(z)$ is bounded on the compact set $|z| \leq \varepsilon$, so $f$ is bounded and entire, so constant, by Liouville’s theorem.

For $f$ meromorphic at $\infty$, $z \to f(1/z)$ has a finite-nosed Laurent expansion at 0, convergent in some punctured neighborhood,

\[
f(1/z) = \frac{c_N}{z^N} + \ldots + c_0 + c_1 z + \ldots \quad \text{(for } 0 < |z| < \varepsilon)\]

On the compact set $|z| \leq 1/\varepsilon$, $f$ itself can have only finitely-many poles, say at $z_1, \ldots, z_n$, of orders $\nu_1, \ldots, \nu_n$. The function $g(z) = (z - z_1)^{\nu_1} \ldots (z - z_n)^{\nu_n} f(z)$ has no poles in $|z| \leq 1/\varepsilon$, and $g(z)$ is meromorphic at $\infty$, since each $(z - z_j)^{\nu_j}$ is meromorphic at $\infty$. Then

\[
g(z) = c_N z^N + \ldots + c_0 + \frac{c_1}{z} + \ldots \quad \text{(for } |z| > 1/\varepsilon)\]

and $z^{-N} g(z)$ is bounded on $|z| > 1/\varepsilon$. The continuous function $|g(z)|$ is certainly bounded on the compact $|z| \leq 1/\varepsilon$, so $|g(z)| \leq B \cdot |z|^N$ for some $B$ and $N$. As in the proof of Liouville’s theorem, an entire function admitting such a bound is a polynomial of degree at most $N$. Thus, the original $f$ was a rational function.

4. Linear fractional (Möbius) transformations

The general linear group $GL_2(\mathbb{C})$ is the group of multiplicatively invertible two-by-two complex matrices. This group acts on two-by-one complex matrices $\mathbb{C}^2$ by matrix multiplication:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix}
\]

The linearity of this action is that $g(c \cdot v) = c \cdot g(v)$ for $g \in GL_2(\mathbb{C})$, $c \in \mathbb{C}$, and $v \in \mathbb{C}^2$. In particular, the action of $GL_2(\mathbb{C})$ stabilizes the equivalence classes $\mathbb{C}^* \cdot v$ used to form $\mathbb{CP}^1$: 

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbb{C}^* = \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix} \cdot \mathbb{C}^*
\]
On the image \((z, 1)\) of a point \(z \in \mathbb{C}\) in \(\mathbb{C}P^1\), in homogeneous coordinates

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  z \\
  1
\end{pmatrix}
= \begin{pmatrix}
  az + b \\
  cz + d
\end{pmatrix}
\]

In the typical case that \(cz + d \neq 0\),

\[
\begin{pmatrix}
  az + b \\
  cz + d
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  \frac{az + b}{cz + d} \\
  1
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  \frac{az + b}{cz + d} \\
  1
\end{pmatrix}
\cdot \mathbb{C}^\infty
\]

That is, the point \(z \in \mathbb{C} \subset \mathbb{C}P^1\) is mapped to \(\frac{az + b}{cz + d} \in \mathbb{C} \subset \mathbb{C}P^1\) when \(cz + d \neq 0\). When \(cz + d = 0\),

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  z \\
  1
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  az + b \\
  0
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  1 \\
  0
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \infty
\]

Write

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}(z)
= \frac{az + b}{cz + d}
\]

with the implicit qualification that the image is \(\infty\) when \(cz + d = 0\).

We can see where the point \(\infty\) is mapped:

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}(\infty)
= \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}\begin{pmatrix}
  1 \\
  0
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  a \\
  c
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  \frac{a}{c} \\
  1
\end{pmatrix}
\cdot \mathbb{C}^\infty
\]

That is,

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}(\infty)
= \begin{pmatrix}
  \frac{a}{c} \\
  1
\end{pmatrix}
\cdot \mathbb{C}^\infty
\]

The continuity of the action of \(GL_2(\mathbb{C})\) on \(\mathbb{C}^2\) results in the continuity of the action of \(GL_2(\mathbb{C})\) on \(\mathbb{C}P^1\).

**[4.1] Remark:** Similarly, \(GL_n(\mathbb{C})\) acts by generalized linear fractional transformations on \(\mathbb{C}P^{n-1}\), by

\[
\begin{pmatrix}
  g_{11} & \cdots & g_{1n} \\
  \vdots & \ddots & \vdots \\
  g_{n1} & \cdots & g_{nn}
\end{pmatrix}
\begin{pmatrix}
  \omega_{11} \\
  \vdots \\
  \omega_n
\end{pmatrix}
\cdot \mathbb{C}^\infty
= \begin{pmatrix}
  g_{11}\omega_{11} + \cdots + g_{1n}\omega_n \\
  \vdots \\
  g_{n1}\omega_{11} + \cdots + g_{nn}\omega_n
\end{pmatrix}
\cdot \mathbb{C}^\infty
\]

**[4.2] Claim:** The holomorphic automorphisms of \(\mathbb{C}P^1\), that is, the meromorphic functions \(f\) on \(\mathbb{C}\) also meromorphic at infinity, and have inverse maps of the same sort, are exactly the linear fractional transformations.

**Proof:** From above, \(f(z) = P(z)/Q(z)\) for polynomials \(P, Q\), with \(Q\) not identically 0. Without loss of generality, we can suppose \(P, Q\) are relatively prime in the (Euclidean) ring \(\mathbb{C}[X]\). If both are constant, then \(f\) is constant, contradicting injectivity.

If \(Q\) has positive degree, then it has a zero \(z_0\), and \(f(z_0) = \infty\). Let \(\gamma\) be a linear fractional transformation mapping \(\infty \rightarrow z_0\). Replacing \(f\) by \(f \circ \gamma\), the modified \(f\) maps \(\infty \rightarrow \infty\). No other point can be mapped to \(\infty\), by injectivity, so this modified \(f\) is be a polynomial.

If the degree of \(f\) is greater than 1 and if \(f\) has two or more distinct complex zeros, it maps those two points to 0, contradicting injectivity. Thus, \(f(z) = c(z - z_0)^n\) for some non-zero \(c\) and for some \(1 \leq n \in \mathbb{Z}\). But this maps \(z_0 + \mu\) to 1 for all \(n^{th}\) roots of unity \(\mu\), contradicting injectivity if \(n > 1\). Thus, the modified \(f\) is linear, and is a linear fractional transformation. Thus, the original \(f\) was a linear fractional transformation.

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