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Counting zeros of $\zeta(s)$

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/complex/notes_2020-21/counting_zeros_of_zeta.pdf]

Zeros of $\zeta(s)$ in the critical strip $0 \leq \text{Re}(s) \leq 1$ are counted\(^1\) using the argument principle and the Laplace-Stirling asymptotic

$$\Gamma(s) = (s - \frac{1}{2}) \log s - s + O(1) \quad \text{(in } \text{Re}(s) \geq \delta > 0 \text{ as } |s| \to \infty)$$

The counting function of interest is\(^2\)

$$N(T) = \text{number of zeros of } \zeta(s) \text{ in } 0 \leq \text{Im}(s) \leq T \text{ and } 0 \leq \text{Re}(s) \leq 1$$

By the argument principle\(^3\)

$$N(T) = \frac{1}{2\pi i} \int_{RT} \frac{\zeta'(s)}{\zeta(s)} \, ds + O(1)$$

where $RT$ is the rectangle connecting $2 \pm iT$ and $-1 \pm iT$, traversed counter-clockwise, deformed slightly to skirting any zeros of $\zeta(s)$. The division by 2 takes into account the double-counting of zeros off the real interval $[0, 1]$, and the $O(1)$ term accommodates miscounting poles at $s = 0, 1$ and any zeros on $[0, 1]$. The proof of the following is simply an estimate of this integral, specifically, giving the leading terms in an asymptotic in $T$.

\[0.1\] Theorem:

$$N(T) = \frac{1}{2\pi} \cdot T \log T - \frac{\log 2\pi e}{2\pi} \cdot T + O(\log T) = \frac{1}{2\pi} T \log \frac{T}{2\pi e} + O(\log T)$$

\[0.2\] Remark: In particular, there are infinitely-many zeros of $\zeta(s)$ in the critical strip.

\[0.3\] Remark: The vertical asymptotics of $\Gamma(s)$ dominate and completely determine the leading terms of the asymptotic expansion, by a direct computation which determines the relevant constants.

Proof: Using the functional equation $\xi(1 - s) = \xi(s)$, and the symmetry $\xi(\bar{s}) = \overline{\xi(s)}$, we integrate only upward from 2 to $2 + iT$, and then left from $2 + iT$ to $\frac{1}{2} + iT$.

The argument-principle integral computes $1/2\pi$ times the net change in the imaginary part of $\log \xi(s)$ over the given paths, requiring continuity of the logarithm. We compute separately the net changes of the imaginary parts of the summands in

$$\log \xi(s) = -\frac{s}{2} \log \pi + \log \Gamma\left(\frac{s}{2}\right) + \log \zeta(s)$$


\[0.2\] The convergent Euler product shows that there are no zeros in $\text{Re}(s) > 1$. The functional equation $\pi^{-\frac{1}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$ shows that the only zeros of $\zeta(s)$ in $\text{Re}(s) < 0$ are where $\Gamma\left(\frac{s}{2}\right)$ has poles, namely, negative even integers. These are the trivial zeros of $\zeta(s)$.

\[0.3\] As usual, $\xi(s)$ is the completed zeta function $\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. The usual notation is $S(T) = \frac{1}{T} \arg \xi(s)$, required to be 0 at $s = 2$, and continuous along the vertical line from 2 to $2 + iT$ and then to $\frac{1}{2} + iT$. When there is a zero along $(0, 1) + iT$, compute $S(T)$ slightly above $T$. 

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Obviously the net change of imaginary part of the logarithm of \( \pi^{-s/2} \) is

\[
\text{Im} \left( \log \pi^{-(s+it)/2} - \log \pi^{-s/2} \right) = \text{Im} \left( -\frac{1}{2} + \frac{1}{2} \log \pi \right) = -\frac{T}{2} \log \pi
\]

From

\[
\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + O(1)
\]

we have

\[
\log \Gamma\left( \frac{s}{2} \right) = \left( \frac{s}{2} - \frac{1}{2} \right) \log \frac{s}{2} - \frac{s}{2} + O(1)
\]

Thus, the net change from 2 to \( \frac{1}{2} + iT \) is

\[
\text{Im} \left( \log \Gamma\left( \frac{\frac{1}{2} + iT}{2} \right) - \log \Gamma\left( \frac{2}{2} \right) \right) = \text{Im} \left( \left( \frac{\frac{1}{2} + iT}{2} - \frac{1}{2} \right) \log \left( \frac{\frac{1}{2} + iT}{2} - \frac{1}{2} \right) + O(1) \right)
\]

\[
= \text{Im} \left( \left( \frac{1}{4} + \frac{iT}{2} \right) \left( \frac{\pi i}{2} + \log \left( \frac{T}{2} + O(1) \right) \right) \right) - \frac{T}{2} + O(1) = \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} + O(1)
\]

Since \( s = 2 + iT \) is nicely within the region of absolute convergence of the Euler product, \( \log \zeta(2 + iT) \) is bounded on that line, so the net change in the imaginary part of the argument of \( \zeta(s) \) from 2 to \( 2 + iT \) is \( O(1) \).

The subtle computation concerns the net change in the argument of \( \zeta(s) \) from \( 2 + iT \) to \( \frac{1}{2} + iT \). We recall a version of part of a relevant lemma from [Titchmarsh/Heath-Brown 1951/1989], page 213, which uses Jensen’s Lemma to approximate the number of zeros, hence, the change in argument, in terms of the growth, of a function:

**[0.4] Lemma**: Let \( f \) be a holomorphic function on a vertical strip \(-1 \leq \sigma \leq 4\), except possibly for a simple pole at \( s = 1 \). Suppose that \( f(\overline{s}) = \overline{f(s)} \). Assume a lower bound \(|f(2 + it)| \geq m > 0\), and a family of upper bounds

\[
|f(\sigma + it)| \leq M(T) \quad \text{for} \quad \frac{1}{4} \leq \sigma \leq 4 \text{ and } 1 \leq t \leq T
\]

Then, for \( T \) not the vertical coordinate of a zero of \( f \), there is the upper bound for change in argument from \( 2 + iT \) to \( \frac{1}{2} + iT \)

\[
| \arg f\left( \frac{1}{2} + iT \right) - \arg f(2 + iT) | \leq \frac{\pi}{\log((2 - \frac{1}{4})/(2 - \frac{1}{2}))} \cdot \left( \log M(T + 2) + \log \frac{1}{m} \right) + \pi
\]

**[0.5] Remark**: Naturally, some of the details are insignificant, being mere artifacts of the proof. At the same time, we give a more specific version of the result than [Titchmarsh/Heath-Brown 1951/1989].

**Proof**: Let \( q \) be the number of vanishings of \( \text{Re} f(\sigma + iT) \) between \( 2 + iT \) and \( \frac{1}{2} + iT \). The vanishings divide the interval into \( q + 1 \) subintervals on each of which either \( \text{Re} f \geq 0 \) or \( \text{Re} f \leq 0 \). In particular, the value of \( f \) stays in either the right or left half-plane, so the \( \arg f \) cannot change more than \( \pi \) in each subinterval. Thus,

\[
| \arg f\left( \frac{1}{2} + iT \right) - \arg f(2 + iT) | \leq (q + 1) \cdot \pi
\]

Using \( f(\overline{s}) = \overline{f(s)} \), the count \( q \) is the number of zeros of \( g(z) = \frac{1}{2} \left[ f(z + iT) + f(z - iT) \right] \) on the real interval \( \frac{1}{2} \leq z \leq 2 \). Certainly this count is bounded by the number of zeros of \( g(z) \) in the disk \( |z - 2| \leq 2 - \frac{1}{2} \).
Let \(\nu(r)\) be the number of zeros in \(|z - 2| \leq r\). Setting up application of Jensen’s lemma,[4] we have an upper bound for \(q\):

\[
\int_0^{2 - \frac{1}{4}} \frac{\nu(r)}{r} \, dr \geq \int_{2 - \frac{1}{2}}^{2 - \frac{1}{4}} \frac{\nu(r)}{r} \, dr \geq \nu(2 - \frac{1}{2}) \cdot \frac{2 - \frac{1}{2}}{2 - \frac{1}{2}} \geq q \cdot \frac{2 - \frac{1}{2}}{2 - \frac{1}{2}}
\]

Jensen’s lemma leads to an upper bound for the integral:

\[
\int_0^{2 - \frac{1}{4}} \frac{\nu(r)}{r} \, dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(2 + (2 - \frac{1}{4})e^{i\theta})| \, d\theta - \log |g(2)| \leq \log M(T + 2) + \log \frac{1}{m}
\]

giving the lemma.

The lemma applies to \(f(s) = \zeta(s)\) since the convergent Euler product is bounded away from 0 on \(2 + i\mathbb{R}\), with bound \(M(t) = t^N\) on a given vertical strip. Thus, the net change in the argument of \(\zeta(s)\) from \(2 + iT\) to \(\frac{1}{2} + iT\) is \(O(\log T)\).

Altogether, the argument principle gives

\[
N(T) = \frac{1}{2} \cdot \frac{1}{2\pi} \cdot 4 \cdot \left( \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} \log \pi \right) + O(\log T)
\]

\[
= \frac{1}{2\pi} \cdot \left( \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} \log \pi \right) + O(\log T) = \frac{T}{2\pi} \log 2 - \frac{T}{2\pi} (1 + \log \pi) + O(\log T)
\]

\[
= \frac{1}{2\pi} \cdot T \log T - \frac{\log 2 \pi e}{2\pi} \cdot T + O(\log T)
\]

which is the asserted asymptotic.

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Bibliography:


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[4] *Jensen’s Lemma* usually appears as follows: for holomorphic \(f\) on \(|z| \leq r\), no zeros on \(|z| = r\), and \(f(0) \neq 0\),

\[
\log |f(0)| - \sum_{\rho} \log \left| \frac{\rho}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \hspace{1cm} \text{(summed over zeros \(|\rho| < r\) of \(f\))}
\]

Letting \(\nu(t)\) be the number of zeros of size less than \(t\),

\[
- \sum_{\rho} \log \left| \frac{\rho}{r} \right| = \sum_{\rho} (\log r - \log |\rho|) = \sum_{\rho} \int_{|\rho|}^{r} \frac{dt}{r} = \int_0^{r} \nu(t) \frac{dt}{t}
\]