14a. Schwarz’ lemma and automorphisms of \( \mathfrak{H} \)

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1. Schwarz’ lemma
2. Holomorphic automorphisms of \( \mathcal{D} \) fixing 0
3. Holomorphic automorphisms of \( \mathfrak{H} \)

Schwarz’ Lemma is a fairly immediate consequence of the maximum modulus principle. Schwarz’ lemma is the key technical point in classification of the holomorphic automorphisms of the upper half-plane \( \mathfrak{H} \) (equivalently, of the open unit disk \( \mathcal{D} \)). Namely, the linear fractional transformations

\[
  z \rightarrow \frac{az + b}{cz + d} \quad \text{(for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})\text{)}
\]

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1. **Schwarz’ lemma**

[1.1] **Theorem**: (Schwarz) Let \( f \) be holomorphic on the open unit disk \( \mathcal{D} \), with \( f(0) = 0 \), and \( |f(z)| \leq 1 \) for all \( z \in \mathcal{D} \). Then \( |f(z)| \leq |z| \) for all \( z \), and \( |f'(0)| \leq 1 \). Further, if either equality holds, then \( f(z) = \alpha \cdot z \) with some \( \alpha \in \mathbb{C}^\times \) with \( |\alpha| = 1 \).

**Proof**: Apply the maximum modulus principle to \( g(z) = f(z)/z \), holomorphic on the disk because \( f(0) = 0 \). For fixed \( z_o \) with \( |z| < 1 \), for any \( r \) satisfying \( |z_o| \leq r < 1 \), the maximum modulus principle gives

\[
  |g(z_o)| \leq \sup_{|z|=r} |g(z)| \leq \sup_{|z|=r} \frac{|f(z)|}{r} \leq 1/r
\]

Since this holds for every such \( r \), in fact \( |g(z)| \leq 1 \). That is, \( |f(z)| \leq |z| \) for \( z \neq 0 \). The limit at \( z = 0 \) gives \( |f'(0)| \leq 1 \).

Further, if \( |f(z_o)| = |z_o| \) for some \( |z_o| < 1 \), then \( g(z_o) = 1 \). Since \( |g(z)| \leq 1 \) throughout \( |z| < 1 \), by the sharp form of the maximum modulus principle, \( g(z) \) is a constant \( \alpha \) with \( |\alpha| = 1 \). That is, \( f(z)/z = \alpha \), and \( f(z) = \alpha \cdot z \), with \( |\alpha| = 1 \).

Similarly, since \( f'(0) = g(0) \), if \( |f'(0)| = 1 \), then \( |g(0)| = 1 \), and the same application of the sharp form of the maximum modulus principle shows that \( f(z) = \alpha \cdot z \) with some \( |\alpha| = 1 \).  

2. **Holomorphic automorphisms of \( \mathcal{D} \) fixing 0**

[2.1] **Corollary**: Let \( f \) be a holomorphic bijection of \( \mathcal{D} \) to itself, with \( f(0) = 0 \). Then \( f^{-1} \) is also holomorphic, and \( f(z) = \alpha \cdot z \) with some \( |\alpha| = 1 \).

**Proof**: For convenience, the lemma below recalls the argument that a bijective holomorphic function has a holomorphic inverse.

Schwarz’ lemma gives \( |f'(0)| \leq 1 \). Since \( f \) is bijective, it is certainly bijective on a neighborhood of 0, so \( f'(0) \neq 0 \). By the holomorphic inverse function theorem, \( f^{-1} \) is holomorphic on a neighborhood of 0, and
(f^{-1})'(0) = 1/f'(0). Assuming f^{-1} is holomorphic on the whole disk, |(f^{-1})'(0)| \leq 1 by Schwarz’ lemma. Thus, |(f^{-1})'(0)| = |f'(0)| = 1. Again by Schwarz’ lemma, f(z) = \alpha \cdot z with some |\alpha| = 1. //

As invoked at the beginning of the proof:

[2.2] Lemma: A bijective holomorphic function g (on a fixed open set) has a holomorphic inverse.

Proof: Holomorphy is a local property. Away from points where \( g'(z_0) = 0 \), the holomorphic inverse function theorem gives a local holomorphic inverse. It would suffice to know that g cannot be bijective locally in a neighborhood of any point \( z_0 \) where \( g'(z_0) = 0 \). A stronger statement is of some interest:

[2.3] Lemma: Let g be holomorphic on a neighborhood of \( z_0 \), and for some \( n \)
\[ g(z_0) = g'(z_0) = g''(z_0) = \ldots = g^{(n-1)}(z_0) \] (n \geq 2)
Then g is locally \( n \)-to-1 on a punctured neighborhood of \( z_0 \), in the following strong sense: There are sufficiently small \( R > 0 \) and \( 0 < r < R \) such that for \( 0 < |w_o| < r \) there are exactly \( n \) points \( z_1, \ldots, z_n \), in \( 0 < |z| \leq R \) such that \( g(z_j) = w_o \). The points \( z_1, \ldots, z_n \) are distinct.

Proof: Without loss of generality \( z_0 = 0 \). Multiplication by a (non-zero) constant also does not affect bijectivity. Thus, near \( z_0 = 0 \),
\[ g(z) = z^n + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots \] (with \( |c_\ell| \leq C \cdot M^\ell \) for some \( C, M \))
For \( 0 < R < \frac{1}{nM} \), the circle \( |z| = R \) is inside the disk of convergence of that power series. For \( |z| \leq R \),
\[ |g(z) - z^n| \leq |z|^n \sum_{\ell \geq n+1} C \cdot M^\ell \cdot R^{\ell-n} = |z|^n \cdot C \cdot \frac{M^{n+1} \cdot R}{1 - MR} \] (in \( |z| \leq R \))
Thus, for \( R < \frac{1 - MR}{C M^{n+1}} \), this is smaller than \( |z|^n \). Thus, in \( 0 < |z| \leq R \), \( |g(z)| \geq |z|^n - |g(z) - z^n| > 0 \). That is, \( g(z) \) does not vanish in \( 0 < |z| \leq R \). Visibly, \( g \) vanishes to order \( n \) at 0. (Rouché’s theorem gives a similar conclusion.) Similarly, further shrinking \( R \) if necessary, \( g'(z) \) is also non-vanishing in \( 0 < |z| < R \).

Let \( r = \min |z|=R |g(z)| \). From the argument principle, for \( |w_o| < r \),
\[ \frac{1}{2\pi i} \int_{|z|=R} \frac{g'(z)}{g(z) - w_o} \, dz = \text{number of times } g \text{ takes value } w_o \text{ inside } |z| = R \]
For \( w_o = 0 \), the value is \( n \). In the region \( |w_o| < r \), the integral is a continuous function of \( w_o \) (in fact, holomorphic). But it can only take integer values. Since \( \{w: |w| < r\} \) is connected, that integral must be constant on \( |w_o| < r \). That is, each such value \( w_o \) is taken exactly \( n \) times in \( |z| \leq R \). Since the derivative \( g' \) is non-vanishing on \( 0 < |z| \leq R \), for \( 0 < |w_o| < r \) the points where \( g \) takes value \( w_o \) must be distinct.

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3. Holomorphic automorphisms of \( \mathcal{H} \)

[3.1] Corollary: The linear fractional transformations \( z \to \frac{az + b}{cz + d} \) with \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \) give all the holomorphic automorphisms of \( \mathcal{H} \).

Proof: [... iou ...]

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