1. Prove that a topological vector space is normable (meaning has a topology given by a norm) if and only if it has a countable local basis (at 0) consisting of bounded open sets \( U_i \) (meaning that for any other open \( V \) containing 0, there exists real \( t_0 \) such that for \( t \geq t_0 \) one has \( U_i \subseteq tV \)).

2. Let \( X \) be a non-compact (normal) topological space. Prove that the completion of \( C_c^\infty(X) \) with the sup-norm is \( C_c^\infty(X) \), the space of continuous functions \( f \) going to 0 at infinity (in the sense that, given \( \varepsilon > 0 \) there is a compact \( K \) such that off \( K \) one has \( |f(x)| < \varepsilon \)).

3. Prove that \( C^\infty[0, b] \) is not normable.

4. Prove that \( C^\infty(\mathbb{R}) \) is not normable.

5. Prove that a topological vector space is metrizable (meaning there’s a metric which engenders the given topology) if and only if it has a countable local basis (at 0).

6. Why can’t \( C^\infty(\mathbb{R}) \) be made into a Frechet space?

7. Let \( X \) be a \( \sigma \)-countable topological space (assumed normal, so that there are sufficiently many continuous functions on it). Show that \( C^\infty(X) \) has a Frechet-space structure.

8. If \( X \) is not \( \sigma \)-countable will \( C^\infty(X) \) have a Frechet-space structure?

9. Let \( \delta : C_c^\infty(\mathbb{R}) \to \mathbb{C} \) be the continuous linear function \( \delta(f) = f(0) \)

Prove that there is no continuous linear functional on \( L^2(\mathbb{R}) \) whose restriction to \( C_c^\infty(\mathbb{R}) \) is \( \delta \).

10. Prove that \( C^\infty(\mathbb{R}) \) is a Frechet space, in particular is complete, with the metric

\[
d(f, g) = \sum_{n=0}^\infty 2^{-n} \frac{\sup_{|x| \leq n} |f(x) - g(x)|}{1 + \sup_{|x| \leq n} |f(x) - g(x)|}
\]

11. Show that the usual product topology on a product \( \prod_{\alpha \in A} X_\alpha \) of topological spaces \( X_\alpha \) does have the mapping property that for every collection \( f_\alpha : W \to X_\alpha \) of continuous maps there is a unique map \( f : W \to \prod_{\alpha} X_\alpha \) such that \( f_\alpha = p_\alpha \circ f \), where \( p_\alpha \) is the projection from the product to \( X_\alpha \). (And \( p_\alpha \) is continuous.)

12. Let \( V \) be a topological vector space over \( \mathbb{C} \) and \( W \) a complex vector subspace which is not topologically closed. Show that the quotient \( V/W \) is a topological vector space in which scalar multiplication and vector addition are continuous, but which is not Hausdorff.

13. Let \( X \) be a vector space with a topology such that vector addition and scalar multiplication are continuous. Define an equivalence relation \( \sim \) on \( X \) by \( x \sim y \) if there are open sets \( U \ni x \) and \( V \ni y \) with \( U \cap V = \emptyset \). Define the Hausdorffization \( X^H \) of \( X \) to be the quotient space \( X/\sim \), with quotient map \( q : X \to X^H \).

(a) Prove that \( \sim \) really is an equivalence relation.

(b) Prove that \( X^H \) is Hausdorff, and \( q : X \to X^H \) is continuous. (c) Prove that for a continuous linear map \( f : X \to Y \) with topological vector space (Hausdorff) \( Y \), there is a unique continuous linear \( f^H : X^H \to Y \) such that \( f = f^H \circ q \).

14. (a) Give an example to show that for more general topological spaces without a vector space structure the definition (just above) of Hausdorffization sufficient for vector spaces fails. (b) As a second try: Let \( X \) be a topological space. Say that two points \( x, y \) in \( X \) are inseparable if there are no open sets \( U \ni x \) and \( V \ni y \) with \( U \cap V = \emptyset \). Define an equivalence relation \( \sim \) on \( X \) by \( x \sim y \) if there are points \( x_1, x_2, \ldots, x_n \) such that \( x_1 = x \) and \( x_n = y \), and \( x_i \) and \( x_{i+1} \) are inseparable for all \( i \). Define the Hausdorffization \( X^H \) of \( X \) to be the quotient space \( X/\sim \), with quotient map \( q : X \to X^H \). Give an example to show that this version of \( X^H \) is not necessarily Hausdorff. (c) Try defining the Hausdorffization \( X^H \) of \( X \) by the condition that there is a continuous \( q : X \to X^H \) and, for a continuous map \( f : X \to Y \) with Hausdorff \( Y \), there is a unique continuous \( f^H : X^H \to Y \) such that \( f = f^H \circ q \).