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Exercises on General TVS's

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1. Prove that a topological vector space is **normable** (meaning has a topology given by a norm) if and only if it has a countable local basis (at 0) consisting of *bounded* open sets U_i (meaning that for any other open V containing 0, there exists real t_o such that for $t \geq t_o$ one has $U_i \subset tV$).

2. Let X be a non-compact (normal) topological space. Prove that the completion of $C_c^o(X)$ with the sup-norm is $C_o^o(X)$, the space of continuous functions f **going to** 0 **at infinity** (in the sense that, given $\varepsilon > 0$ there is a compact K such that off K one has $|f(x)| < \varepsilon$).

3. Prove that $C^\infty[a, b]$ is not normable.

4. Prove that $C^o(\mathbf{R})$ is not normable.

5. Prove that a topological vector space is **metrizable** (meaning there's a metric which engenders the given topology) if and only if it has a countable local basis (at 0).

6. Why can't $C_c^o(\mathbf{R})$ be made into a Frechét space?

7. Let X be a σ-countable topological space (assumed *normal*, so that there are sufficiently many continuous functions on it). Show that $C^o(X)$ has a Frechét-space structure.

8. If X is *not* σ-countable will $C^o(X)$ have a Frechét-space structure?

9. Let $\delta : C_c^o(\mathbf{R}) \to \mathbf{C}$ be the continuous linear function

$$\delta(f) = f(0)$$

Prove that there is no continuous linear functional on $L^2(\mathbf{R})$ whose restriction to $C_c^o(\mathbf{R})$ is δ.

10. Prove that $C^o(\mathbf{R})$ is a Frechét space, in particular is *complete*, with the metric

$$d(f, g) = \sum_{n=0}^{\infty} 2^{-n} \, \frac{\sup_{|x| \leq n} |f(x) - g(x)|}{1 + \sup_{|x| \leq n} |f(x) - g(x)|}$$

11. Show that the usual product topology on a product $\prod_{\alpha \in A} X_a lf$ of topological spaces X_α does have the mapping property that for every collection $f_\alpha : W \to X_\alpha$ of continuous maps there is a unique map $f : W \to \prod_\alpha X_\alpha$ such that $f_\alpha = \mathbf{p}_\alpha \circ f$, where p_α is the projection from the product to X_α. (And p_α is continuous.)

12. Let V be a topological vector space over $\mathbf{C}$ and W a complex vector subspace which is *not* topologically closed. Show that the quotient V/W is a complex topological vector space in which scalar multiplication and vector addition are continuous, but which is *not* Hausdorff.

13. Let X be a vector space with a topology such that vector addition and scalar multiplication are continuous. Define an equivalence relation $\sim$ on X by $x \sim y$ if there are *no* open sets $U \ni x$ and $V \ni y$ with $U \cap V = \phi$. Define the **Hausdorffization** X^H of X to be the quotient space $X/\sim$, with quotient map $q : X \to X^H$.
(a) Prove that $\sim$ really is an equivalence relation.
(b) Prove that X^H is Hausdorff, and $q : X \to X^H$ is continuous. (c) Prove that for a continuous linear map $f : X \to Y$ with topological vector space (Hausdorff) Y, there is a unique continuous linear $f^H : X^H \to Y$ such that $f = f^H \circ q$.

14. (a) Give an example to show that for more general topological spaces without a vector space structure the definition (just above) of Hausdorffization sufficient for vector spaces *fails*. (b) As a second try: Let X be a topological space. Say that two points x, y in X are **inseparable** if there are *no* open sets $U \ni x$ and $V \ni y$ with $U \cap V = \phi$. Define an equivalence relation $\sim$ on X by $x \sim y$ if there are points $x_1, x_2, \ldots, x_n$ such that $x_1 = x$ and $x_n = y$, and x_i and x_{i+1} are inseparable for all i. Define the **Hausdorffization** X^H of X to be the quotient space $X/\sim$, with quotient map $q : X \to X^H$. Give an example to show that *this* version of X^H is not necessarily Hausdorff. (c) Try defining the Hausdorffization X^H of X by the condition that there is a continuous $q : X \to X^H$ and, for a continuous map $f : X \to Y$ with Hausdorff Y, there is a unique continuous $f^H : X^H \to Y$ such that $f = f^H \circ q$.