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Exercises on distributions

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1. Give a linear functional on $L^2(\mathbf{R})$ which is not continuous.

2. Prove in detail that $s \to u_s$ where u_s is integrate-against $|x|^s$ on $\mathbf{R}^n$ is *complex differentiable* as a $\mathcal{D}'$-valued function.

3. By considering $u_s =$ (integrate-against)$|x|^s \log |x|$, show that

$$\Delta \log |x| = c \cdot \delta$$

for a constant which you should determine.

4. Using coordinates $x + iy = z$ on $\mathbf{R}^2 \approx \mathbf{C}$, and letting

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

and considering $u_s = |z|^s$ show that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z} = c \cdot \delta$$

for a constant which you should determine.

5. Prove that the rotation-invariant distributions supported at $\{0\}$ are $\Delta^\ell \delta$ for integers $\ell \geq 0$.

6. To understand the (non-normalized) Hilbert transform

$$Hf(x) = \int_{-\infty}^{\infty} \frac{f(y)}{x - y} \, dy$$

consider $u_s = \frac{1}{x} \cdot |x|^s$. Show that u_s has a meromorphic continuation, and is *holomorphic* at $s = 0$. Explain how this would allow us to make sense of H without mentioning Cauchy Principle Values, etc.

7. Is

$$f \to \int_{-\infty}^{\infty} e^{ie^x} f(x) \, dx$$

a tempered distribution?