We review the basic terminology concerning metric spaces, and prove the very important Baire category theorem, for both complete metric spaces and locally compact Hausdorff spaces.

1. Metric spaces, completeness

Recall that a metric space $X, d$ is a set $X$ with a metric $d(\cdot)$, a real-valued function such that, for $x, y, z \in X$,
- (Positivity) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
- (Symmetry) $d(x, y) = d(y, x)$
- (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ A metric space $X$ has a natural topology with basis given by open balls
  \[ \{ y \in X : d(x, y) < r \} \]

of radius $r > 0$ centered at points $x \in X$

A Cauchy sequence in a metric space $X$ is a sequence $x_1, x_2, \ldots$ with the property that for every $\varepsilon > 0$ there is $N$ sufficiently large such that for $i, j \geq N$ we have $d(x_i, x_j) < \varepsilon$. A point $x \in X$ is a limit of that Cauchy sequence if for every $\varepsilon > 0$ there is $N$ sufficiently large such that for $i \geq N$ we have $d(x_i, x) < \varepsilon$. A subset $X$ of a metric space $Y$ is dense in $Y$ if every point in $Y$ is a limit of a Cauchy sequence in $X$.

The following standard lemma is often useful, and makes explicit a bit of intuition.

**Lemma:** Let $\{x_i\}$ be a Cauchy sequence in a metric space $X, d$, and suppose that the sequence converges to $x$ in $X$. Given $\varepsilon > 0$, let $N$ be sufficiently large such that for $i, j \geq N$ we have $d(x_i, x_j) < \varepsilon$. Then for $i \geq N$ we also have $d(x_i, x) \leq \varepsilon$.

**Proof:** Let $\delta > 0$ and take $j \geq N$ also large enough such that $d(x_j, x) < \delta$. Then for $i \geq N$ by the triangle inequality
\[ d(x_i, x) \leq d(x_i, x_j) + d(x_j, x) < \varepsilon + \delta \]
Since this holds for every $\delta > 0$ we have the result. \(/\!\!/\)

A metric space is **complete** if every Cauchy sequence has a limit.\footnote{Recall that a topological vector space is **locally compact** if every point has an open neighborhood with compact closure. A space is **Hausdorff** if for any two points $x, y$ there are opens $U, V$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.}

2. Completions

\footnote{Convergence of Cauchy sequences is more properly called **sequential completeness**. In fact, for metric spaces, sequential completeness implies the strongest form of completeness, namely convergence of Cauchy nets, as we will observe more carefully later. This is **not** so important at the moment, but will have some importance for non-metrizable spaces, which are **rarely** complete (in the strongest sense), but in practice often are at least sequentially complete. A useful form of completeness stronger than sequential completeness but weaker than outright completeness is **local completeness**, also called **quasi-completeness**, which will play a significant role later.}
A map $f : X \to Y$ from one metric space $X$, $d_X$ to another $Y$, $d_Y$ is an isometry if it is distance-preserving, that is, if
$$d_Y(f(x), f(x')) = d_X(x, x')$$
for all $x, x' \in X$. Certainly an isometry is continuous.

The usual definition of the completion $Y$ of a metric space $X$ is that $Y$ is a complete metric space with an isometry $i : X \to Y$ such that the image $i(X)$ is dense.\(^3\)

Before describing any construction of a completion, we can prove some things about the behavior of any possible completion. In particular, we will prove that any two completions are naturally isometric to each other. Thus, whatever choice of construction we make the outcome will be the same.

**Proposition:** Let $i : X \to Y$ and $j : X \to Z$ be two completions of a metric space $X$. Then there is a unique bijective isometry $h : Y \to Z$ such that
$$j = h \circ i$$

**Proof:** Given $y \in Y$, choose a Cauchy sequence $x_k$ in $X$ such that $i(x_k)$ converges to $y$, and try to define
$$h(y) = \lim_{k} j(x_k)$$

Even though we may anticipate that this will work fine, it is not a priori clear that the limit exists, that it is well-defined, etc. Although nothing surprising happens, we check those details, as follows.

Since the map $j$ preserves distances, the sequence $j(x_k)$ is Cauchy in $Z$, so has a limit since $Z$ is complete. For well-definedness, for $x_k$ and $x'_k$ two Cauchy sequences whose images $i(x_k)$ and $i(x'_k)$ approach $y$, since $i$ is an isometry eventually $x_k$ is close to $x'_k$. Thus, $j(x_k)$ is close to $j(x'_k)$ by continuity. Thus, $h(y)$ is well-defined.

To show that $h$ is an isometry, let $y, y' \in Y$, with two Cauchy sequences $x_t$ and $x'_t$ approaching $y$ and $y'$ respectively. Given $\varepsilon > 0$, let $N$ be large enough such that for $r, s \geq N$ we have $d_Z(h(i(x_r)), h(i(x_s))) < \varepsilon$ and $d_Z(h(i(x'_r)), h(i(x'_s))) < \varepsilon$ where $d_Z(\cdot, \cdot)$ is the metric in $Z$. Then (from the lemma above!) for such $r$ also
$$d_Z(h(i(x_r)), h(y)) \leq \varepsilon$$

and
$$d_Z(h(i(x'_r)), h(y')) \leq \varepsilon$$

By the triangle inequality
$$d_Z(h(y), h(y')) \leq d_Z(h(y), h(i(x_r))) + d_Z(h(i(x_r)), h(i(x'_r))) + d_Z(h(i(x'_r)), h(y')) \leq \varepsilon + d(x_r, x'_r) + \varepsilon$$
since $j = h \circ i$ is an isometry $X \to Z$. But also, letting $d_Y(\cdot, \cdot)$ be the metric on $Y$,
$$d_Y(i(x_r), i(x'_r)) \leq d_Y(i(x_r), y) + d_Y(y, y') + d_Y(i(x'_r), y')$$

and
$$d_Y(i(x_r), i(x'_r)) \geq -d_Y(i(x_r), y) + d_Y(y, y') - d_Y(i(x'_r), y')$$

so
$$|d(x_r, x'_r) - d_Y(y, y')| \leq 2\varepsilon$$

Thus
$$d_Z(h(y), h(y')) \leq d_Y(y, y') + 4\varepsilon$$

\(^3\) The usual discussion of completion thus may accidentally neglect questions of uniqueness.
This proves that \( h : Y \to Z \) is an isometry. In particular, it is injective.

Now claim that \( h : Y \to Z \) is a surjection. Indeed, if \( j(x_k) \) is a Cauchy sequence approaching a point \( z \in Z \), then \( x_k \) is Cauchy in \( X \) since \( j \) is an isometry. Then \( i(x_k) \) is Cauchy in \( Y \) with some limit \( y \), and \( h(y) = z \) by the definition of \( h \). In summary, the natural definition

\[
h(\lim_k i(x_k)) = \lim_k j(x_k)
\]

gives a bijective isometry from the one completion to the other.

Now we give the standard construction of a completion of \( X \). Let \( C \) be the collection of Cauchy sequences in \( X \). Let \( \approx \) be the relation on Cauchy sequences defined by \( \{x_s\} \sim \{y_t\} \) if and only if for every \( \varepsilon > 0 \) there is \( N \) sufficiently large such that for \( r, s \geq N \) we have \( d(x_r, y_s) < \varepsilon \). Attempt to define a metric \( D \) on \( C/ \sim \) by

\[
D(\{x_s\}, \{y_t\}) = \lim_s d(x_s, y_t)
\]

We must verify that this is well-defined on the quotient \( C/ \sim \) and gives a metric. We have an injection \( i : X \to C/ \sim \) by

\[
x \mapsto \{x, x, x, \ldots\} \mod \sim
\]

We should prove that this is an isometry, and that \( C/ \sim \) really is complete.

### 3. The Baire category theorem

This standard result is both indispensable and mysterious.

A set \( E \) in a topological space \( X \) is **nowhere dense** if its closure \( \overline{E} \) contains no non-empty open set. A **countable union** of nowhere dense sets is said to be of **first category**, while every other subset (if any) is of **second category**. The idea (not at all clear from this traditional terminology) is that first category sets are **small**, while second category sets are **large**. In this terminology, the theorem’s assertion is equivalent to the assertion that (non-empty) complete metric spaces and locally compact Hausdorff spaces are of **second category**.

Further, a \( G_\delta \) set is a countable intersection of open sets. Concommitantly, an \( F_\sigma \) set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, a **countable intersection of dense \( G_\delta \)’s is still a dense \( G_\delta \).**

**Theorem: (Baire category)** Let \( X \) be a set with metric \( d \) making \( X \) a **complete metric space**. Or let \( X \) be a locally compact Hausdorff topological space. The intersection of a **countable** collection \( U_1, U_2, \ldots \) of **dense** open subsets \( U_i \) of \( X \) is still **dense** in \( X \).

**Proof:** Let \( B_o \) be a non-empty open set in \( X \), and show that \( \bigcap_i U_i \) meets \( B_o \). Suppose that we have inductively chosen an open ball \( B_{n-1} \). By the denseness of \( U_n \), there is an open ball \( B_n \) whose closure \( \overline{B_n} \) satisfies

\[
\overline{B_n} \subset B_{n-1} \cap U_n
\]

Further, for complete metric spaces, take \( B_n \) to have radius less than \( 1/n \) (or any other sequence of reals going to \( 0 \)), and in the locally compact Hausdorff case take \( B_n \) to have compact closure.

Let

\[
K = \bigcap_{n \geq 1} \overline{B_n} \subset B_o \cap \bigcap_{n \geq 1} U_n
\]

For complete metric spaces, the centers of the nested balls \( B_n \) form a Cauchy sequence (since they are nested and the radii go to \( 0 \)). By completeness, this Cauchy sequence **converges**, and the limit point lies inside each
closure $\overline{B_n}$, so lies in the intersection. In particular, $K$ is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so $K$ is non-empty, and $B_o$ necessarily meets the intersection of the $U_n$. ///