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Banach and Fréchet spaces of functions

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Many familiar and useful spaces of continuous or differentiable functions are Hilbert or Banach spaces, with pleasant completeness properties, but many are not. Some are Fréchet spaces, thus still complete, but lacking some of the conveniences of Banach spaces. Some other important spaces are not Fréchet, either. Still, some of these important spaces are colimits of Fréchet spaces (or of Banach spaces), and the consequent quasi-completeness suffices for many subsequent applications. Here we look at some naturally occurring Banach and Fréchet spaces. Our main point will be to prove *completeness* with the natural metrics.

All vector spaces are over the complex numbers $\mathbb{C}$, or possibly over the real numbers $\mathbb{R}$, but usually this will not matter.

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1. *Examples of spaces of functions*

Our first examples involve continuous and continuously differentiable functions, $\mathrm{C}^o(K)$ and $C^k[a,b]$. The second sort involves measurable functions with integral conditions, the spaces $L^p(X,\mu)$.

In the case of the natural function spaces, the immediate goal is to give the vector space of functions a metric (if possible) which makes it *complete*, so that we can *take limits* and be sure to stay in the same class of functions. For example, *pointwise* limits of continuous functions can easily fail to be continuous.

[1.0.1] Theorem: The set $\mathrm{C}^o(K)$ of (complex-valued) continuous functions on a compact set K is *complete* when given the metric

$$d(f,g) = |f-g|$$

where $|\,|$ is the *norm*

$$|f|_\infty = |f|_{\mathrm{C}^o} = \sup_{x\in K} |f(x)|$$

Proof: This is a typical three-epsilon argument. The point is *completeness*, namely that a Cauchy sequence of continuous functions has a *pointwise* limit which is a continuous function. First we observe that a Cauchy sequence f_i has a pointwise limit. Given $\varepsilon > 0$, choose N large enough such that for $i,j \geq N$ we have $|f_i - f_j| < \varepsilon$. Then for any x in K $|f_i(x) - f_j(x)| < \varepsilon$. Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \geq N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. Then for $i \geq N$

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \varepsilon + \varepsilon'$$

Since this is true for every positive ε' we have

$$|f_i(x) - f(x)| \leq \varepsilon$$

for every x in K. (That is, the pointwise limit is approached uniformly in x.)

Now we prove that $f(x)$ is continuous. Given $\varepsilon > 0$, let N be large enough so that for $i,j \geq N$ we have $|f_i - f_j| < \varepsilon$. From the previous paragraph

$$|f_i(x) - f(x)| \leq \varepsilon$$

for every x and for $i \geq N$. Fix $i \geq N$ and $x \in K$, and choose a small enough neigborhood U of x such that $|f_i(x) - f_i(y)| < \varepsilon$ for any y in U. Then

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \leq \varepsilon + |f_i(x) - f_i(y)| + \varepsilon < \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit f is continuous at every x in U. ///

As usual, a real-valued or complex-valued function f on a closed interval $[a, b] \subset \mathbb{R}$ is **continuously differentiable** if it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists for all $x \in [a, b]$, and the function $f'(x)$ is in $C^o[a, b]$. Let $C^k[a, b]$ be the collection of k-times continuously differentiable functions on $[a, b]$, with the C^k**-norm**

$$|f|_{C^k} = \sum_{0 \leq i \leq k} \sup_{x \in [a,b]} |f^{(i)}(x)| = \sum_{0 \leq i \leq k} |f^{(i)}|_\infty$$

where $f^{(i)}$ is the i^{th} derivative of f. The **associated metric** on $C^k[a, b]$ is

$$d(f, g) = |f - g|_{C^k}$$

[1.0.2] **Theorem:** The metric space $C^k[a, b]$ is complete.

Proof: The case $k = 1$ already illustrates the key point. As in the case of C^o just above, for a Cauchy sequence f_n in $C^1[a, b]$ the pointwise limits

$$f(x) = \lim_n f_n(x) \qquad g(x) = \lim_n f'_n(x)$$

exist, are approached uniformly in x, and are continuous functions. We must show that f is continuously differentiable by showing that $g = f'$.

By the fundamental theorem of calculus, for any index i, since f_i is continuous,[1]

$$f_i(x) - f_i(a) = \int_a^x f'_i(t)\, dt$$

By an easy form of the Dominated Convergence Theorem[2]

$$\lim_i \int_a^x f'_i(t)\, dt = \int_a^x \lim_i f'_i(t)\, dt = \int_a^x g(t)\, dt$$

Thus

$$f(x) - f(a) = \int_a^x g(t)\, dt$$

from which $f' = g$. ///

[1] This invocation of the fundamental theorem of calculus for integrals of *continuous* functions needs only the very simplest notion of an integral.

[2] It may seem to be overkill to invoke the Dominated Convergence Theorem in this context, but attention to such details helps us avoid many of the gaffes of early 19th-century analysis.

In a somewhat different vein, for $1 \leq p < \infty$, on a measure space[3] (X, μ) with positive measure μ we have the usual L^p spaces

$$L^p(X, \mu) = \{\text{measurable } f : |f|_p < \infty\} \text{ modulo } \sim$$

with the usual L^p norm

$$|f|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

with associated metric

$$d(f, g) = |f - g|_p$$

and where we take a quotient by the equivalence relation

$$f \sim g \text{ if } f - g \text{ is 0 off a set of measure 0}$$

In the special case that $X = \{1, 2, 3, \ldots\}$ with counting measure μ, the L^p-space has a different standard notation

$$\ell^p = \{\text{complex sequences } \{c_i\} \text{ with } \left(\sum_i |c_i|^p \right)^{1/p} < \infty\}$$

[1.0.3] **Theorem:** With $1 \leq p < \infty$ the spaces $L^p(X, \mu)$ are complete metric spaces. Further, for a Cauchy sequence f_i in $L^p(X, \mu)$ there is a subsequence converging *pointwise* off a set of measure 0 in X.

Proof: The triangle inequality here is Minkowski's inequality. To prove completeness and the assertion about the subsequence converging pointwise almost everywhere, choose a subsequence f_{n_i} such that

$$|f_{n_i} - f_{n_{i+1}}|_p < 2^{-i}$$

Then let

$$g_n(x) = \sum_{1 \leq i \leq n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and put

$$g(x) = \sum_{1 \leq i < \infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

Note that the infinite sum is not necessarily claimed to converge to a finite sum for every x. The triangle inequality shows that $|g_n|_p \leq 1$. Fatou's Lemma (itself following from Lebesgue's Monotone Convergence Theorem) asserts that for $[0, \infty]$-valued measurable functions h_i

$$\int_X \left(\liminf_i h_i \right) \leq \liminf_i \int_X h_i$$

Thus, $|g|_p \leq 1$. Thus, $|g|_p$ is finite. Thus,

$$f_{n_1}(x) + \sum_{i \geq 1} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges for almost all $x \in X$. Let $f(x)$ be the sum at points x where the series converges, and on the measure-zero set where the series does not converge let $f(x) = 0$. Certainly

$$f(x) = \lim_i f_{n_i}(x) \quad \text{(for almost all } x\text{)}$$

[3] The space X on which the measure and the functions live need not be a *topological* space, and the measure μ need have no connection with continuity, for this to make sense.

Now prove that this almost-everywhere pointwise limit is the L^p-limit of the original sequence. For $\varepsilon > 0$ take N such that $|f_m - f_n|_p < \varepsilon$ for $m, n \geq N$. Fatou's lemma gives

$$\int |f - f_n|^p \leq \liminf_i \int |f_{n_i} - f_n|^p \leq \varepsilon^p$$

Thus $f - f_n$ is in L^p and hence f is in L^p. And $|f - f_n|_p \to 0$. ///

Of course we are often interested in situations where a measure *does* have a connection with a topology. For X a topological space and a regular positive *Borel* measure finite on compact sets (sometimes called a **Radon measure**), for locally compact X we can put the L^p norm on spaces $C^o(X)$ without taking a quotient. Recall

[1.0.4] Theorem: For a locally compact Hausdorff topological space X with positive regular Borel measure μ finite on compacta, the space $C^o_c(X)$ of compactly-supported continuous functions is *dense* in $L^p(X, \mu)$. ///

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2. *Normed spaces, Banach spaces*

The examples above are several of the standard examples of *normed spaces*, defined below. In fact, those examples were seen to be *complete* as metric spaces, so are *Banach spaces*, as defined below.

A **normed space** or **pre-Banach** is a vector space V with a non-negative real-valued function $|\,|$ (the **norm**) on it with the properties
- *(Positivity)* $|v| \geq 0$ and $|v| = 0$ only for $v = 0$.
- *(Homogeneity)* $|c \cdot v| = |c| \cdot |v|$ for $c \in \mathbb{C}$ and $v \in V$, where $|c|$ is the usual complex absolute value.
- *(Triangle inequality)* For $v, w \in V$, $|v + w| \leq |v| + |w|$.

When V has a norm $|\,|$, there is naturally associated to it a **metric**

$$d(v, w) = |v - w|$$

The positivity of the norm assures that $d(v, w) = 0$ implies $v = w$. The homogeneity implies the symmetry

$$d(v, w) = |v - w| = |-(v - w)| = |w - v| = d(w, v)$$

And the triangle inequality for the norm implies the triangle inequality for the metric

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$$

Further, because of the way it's defined, such a metric is **translation-invariant** meaning that

$$d(x, y) = d(x + z, y + z)$$

A normed space V which is *complete* with the associated metric is said to be a **Banach space**.

Many of the standard examples of naturally normed spaces are in fact complete, though this may require some proof. Two very important examples are

$$\begin{cases} C^o(X), \text{ with sup norm, is a Banach space, for compact } X \\ \\ C^k[a, b], \text{ with } C^k\text{-norm, is a Banach space, for } -\infty < a < b < +\infty \end{cases}$$

[2.0.1] **Remark:** *Hilbert spaces* are Banach spaces, although proof of the triangle inequality requires invocation of the Cauchy-Schwarz-Bunyakowsky inequality. This will be clear in our later discussion of Hilbert spaces.

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3. *More examples of spaces of functions*

For a non-compact topological space X, the space $C^o(X)$ of continuous functions on X is *not* a Banach space. Certainly the sup of the absolute value of a continuous function may be $+\infty$. But, $C^o(X)$ may nevertheless have a complete metric structure under some mild hypotheses on X.

Note that *norms* always produce *metrics*, but *not* every metric arises from a norm.

[3.0.1] **Proposition:** Suppose that X is a countable union of compact subsets K_i, where K_{i+1} contains K_i in its interior. [4] Define *semi-norms*[5]

$$\mu_i(f) = \sup_{x \in K_i} |f(x)|$$

and define a metric

$$d(f,g) = \sum_i 2^{-i} \frac{\mu_i(f-g)}{\mu_i(f-g)+1}$$

Then $C^o(X)$ is a complete metric space.

[3.0.2] **Remark:** If the whole space X is not a *countable* union of compacts, then we cannot form a metric by this procedure. [6]

Proof: If $f \neq g$ then $f(x) \neq g(x)$ for some point x, which lies in some compact K_i, so $d(f,g) > 0$. This proves the positivity property of $d(,)$. The symmetry and triangle inequality follow from the analogous properties of the absolute value of complex numbers.

To prove completeness, observe that (as above) for a Cauchy sequence f_n on each compact K_i the pointwise limit $f(x) = \lim_i f_i(x)$ is a *uniform* limit of continuous functions, so is continuous. Thus, for any point $x \in X$, choose a compact K_i large enough to contain a neighborhood of x, and conclude that the pointwise limit f is continuous at x. ///

[3.0.3] **Remark:** Further, the topology is **locally convex** in the sense that open balls are convex. [7]

[3.0.4] **Remark:** A negative aspect of the metric so constructed is that it contains details that are both irrelevant and capricious. For example, any alternative metric

$$d_{\text{alt}}(f,g) = \sum_i c_i \cdot \frac{\mu_i(f-g)}{\mu_i(f-g)+1}$$

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[4] If we anticipate the *Baire Category Theorem*, then we can more simply require that X be locally compact, Hausdorff, and be a countable union of compacts. (The last condition is σ-compactness.) Then it *follows* that, for given $x \in X$, there is some K_i containing a neighborhood of x.

[5] These are *semi-norms* rather than *norms* since they are not generally positive for non-zero f.

[6] One might recall that an *uncountable* sum of positive real numbers cannot converge.

[7] The sense of *convexity* is as usual. That is, given f, given a radius $r > 0$, and given g, h such that $d(f,g) < \varepsilon$ and $d(f,h) < \varepsilon$, for any $0 \leq t \leq 1$, the linear combination $F = tg + (1-t)h$ also satisfies $d(f,F) < \varepsilon$.

with constants $c_i > 0$ with

$$\sum_i c_i < +\infty$$

has the same desired properties, but obviously assumes different values. That is, the salient properties reside in the *topology* rather than in any particular (among many possible) metric giving that topology.

The space $C^\infty[a,b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a,b]$ in $\mathbb{R}$ is not a Banach space. [8] However, we have

[3.0.5] Proposition: With seminorms

$$\mu_k(f) = \sup_{0 \le i \le k} \sup_{x \in [a,b]} |f^{(i)}(x)|$$

and metric

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\mu_k(f-g)}{\mu_k(f-g)+1}$$

$C^\infty([a,b])$ is complete.

Proof: For f_i a Cauchy sequence in $C^\infty[a,b]$ with this metric, the f_i also form a Cauchy sequence in the Banach space $C^k[a,b]$. Thus, as we saw earlier, the sequence of restrictions converges in $C^k[a,b]$ to a C^k function on $[a,b]$, for every k. ///

[3.0.6] Remark: The same ideas can be used to prove that $C^k(U)$ is Fréchet, for arbitrary open subsets of $\mathbb{R}^n$, and that $C^\infty(U)$ is likewise Fréchet.

[3.0.7] Remark: All the spaces $C^k(\mathbb{R})$ and, more generally, $C^k(U)$, are readily shown to be *locally convex*.

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4. *Fréchet spaces abstractly*

Vector spaces which are complete metrizable, though not Banach, such as $C^o(X)$ with non-compact X, occur often and are important. With some technical additions, these are **Fréchet spaces**, defined just below.

Let V be a complex vector space with a metric $d(,)$. Suppose that d is **translation invariant** in the sense that

$$d(x+z, y+z) = d(x,y)$$

for all $x, y, z \in V$. (Note that this property does hold for the metrics induced from norms.) We give V the topology induced by the metric. A local basis at $v \in V$ consists of open balls centered at v

$$\{w \in V : d(v,w) < r\}$$

The translation invariance implies that the open balls centered at v are none other than the translates

$$v + B_r = \{v + b : b \in B_r\}$$

where B_r is the open ball of radius r centered at 0. That is, because of the translation invariance, the topology at 0 determines the topology on the whole vector space.

[8] It is not completely trivial to prove that it is impossible to give the thing any Banach spaces structure, but we won't worry about this right now.

A topology on V is **locally convex** if there is a local basis at 0 (hence, at every point) consisting of convex [9] sets. A vector space V with a translation-invariant metric $d(,)$ is a **pre-Fréchet space** if the topology is locally convex. If, further, the metric is *complete*, then the space V is a **Fréchet space**. In light of what we proved about the examples above,

$$\begin{cases} C^o(X), \text{ for } \sigma\text{-compact locally compact } X, \text{ is Fréchet} \\ \\ C^k(\mathbb{R}) \text{ is Fréchet} \end{cases}$$

[4.0.1] **Remark:** The local convexity requirement may seem obscure, but does hold in many important cases, such as $C^o(X)$ and $C^k(\mathbb{R})$ treated above, and is crucial for application. But not every metrizable vectorspace is locally convex. The simplest example [10] is the following:

[4.0.2] **Example:** Let $0 < p < 1$. Define

$$|\{c_i\}|_p = \sum_i |c_i|^p$$

This is *not* a norm, because it is not *homogeneous*, since we do not take a p^{th} root. Define

$$\ell^p = \{\{c_i\} : |\{c_i\}|_p < \infty\}$$

For two sequences $v, w \in \ell^p$ define

$$d(v, w) = |v - w|_p$$

With $0 < p < 1$ the space ℓ^p is a *not*-locally-convex metric space.

[9] Again, a set E in a vector space V is **convex** if for any $x, y \in E$ and for $t \in [0, t]$ the *convex combination* $tx + (1 - t)y$ again lies in E.

[10] It may be that the importance of this example is just as a caution against believing that all metrizable vectorspaces are locally convex.