(*February 19, 2005*)

Applications to Fourier series

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

We showed that the Fourier series of periodic C^1 functions f on $\mathbf{R}^1$ converge *uniformly* (that is, in the C^o metric), and also converge *pointwise* to the original f. Thus, in the one dimensional case, the Fourier series of periodic C^1 functions do converge to the functions.

From the general discussion of L^2 functions, we know that a sequence of L^2 functions has a *subsequence* which converges *pointwise*. (In fact, that is how the limit is constructed in proving completeness of L^2.) This applies to Fourier series, but pointedly does *not* say anything about the pointwise convergence of the whole sequence of partial sums.

Theorem: There is a periodic continuous function f on $[0,1]$ (periodicity meaning that $f(0) = f(1)$) such that the Fourier series

$$\sum_{n\in\mathbf{Z}} \hat{f}(n)\, e^{2\pi inx}$$

of f *does not converge* at 0, where as usual

$$\hat{f}(n) = \int_0^1 e^{-2\pi inx}\, f(x)\, dx$$

The Banach-Steinhaus (*Uniform Boundedness*) theorem for Banach spaces V has as a special case that for a collection of continuous linear functionals $\{\lambda_\alpha : \alpha \in A\}$ on V either there is a uniform bound M such that $|\lambda_\alpha| \le M$ for all $\alpha \in A$, or else there is v in the unit ball of V such that

$$\sup_{\alpha\in A} |\lambda_\alpha\, v| = +\infty$$

In fact, the collection of such v is *dense* in the unit ball, and is an intersection of a *countable* collection of dense open sets (called a G_δ).

Consider the functionals

$$\lambda_N(f) = \sum_{|n|\le N} \hat{f}(n)$$

These are partial sums of the Fourier series of f evaluated at 0. Since

$$|\lambda_N(f)| \le \int_0^1 \left| \sum_{|n|\le N} e^{-2\pi inx} \right| |f(x)|\, dx$$

$$\le |f|_\infty \cdot \int_0^1 \left| \sum_{|n|\le N} e^{-2\pi inx} \right| dx = |f|_\infty \cdot |\sum_{|n|\le N} e^{-2\pi inx}|_1$$

The point is to show, rather, that equality holds, namely

$$|\lambda_N| = |\sum_{|n|\le N} e^{-2\pi inx}|_1$$

and that as $n \to \infty$ the latter L^1-norms go to ∞.

First, summing the finite geometric series and doing a little adjustment gives

$$\sum_{|n|\le N} e^{-2\pi inx} = \frac{e^{-2\pi iNx} - e^{-2\pi i(-N-1)x}}{e^{-2\pi ix} - 1} = \frac{e^{2\pi i(N+\frac{1}{2})x} - e^{-2\pi i(N+\frac{1}{2})x}}{e^{\pi ix} - e^{-\pi ix}} = \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}$$

Applying to this the elementary inequality

$$|\sin t| \le |t|$$

gives

$$\int_0^1 \left| \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x} \right| dx \ge \int_0^1 |\sin 2\pi(N+\frac{1}{2})x| \, \frac{1}{x}\, dx = \int_0^{2\pi(N+\frac{1}{2})} |\sin x| \, \frac{1}{x}\, dx$$

$$\ge \sum_{\ell=1}^N \frac{1}{\ell} \int_{2\pi(\ell-1)}^{2\pi\ell} |\sin x| \, dx \ge \sum_{\ell=1}^N \frac{1}{\ell} \to \infty$$

as $N \to \infty$. Thus, the L^1-norms do go to ∞.

Let $g(x)$ be the *sign* of the kernel

$$\sum_{|n|\le N} e^{-2\pi i n x} = \frac{\sin 2\pi(N+\frac{1}{2})x}{\sin \pi x}$$

from above. Let g_j be a sequence of periodic continuous functions with $|g_j| \le 1$ and going to g pointwise. Then by the dominated convergence theorem

$$\lim_j \lambda_N(g_j) = \lim_j \int_0^1 g_j(x) \sum_{|n|\le N} e^{-2\pi i n x}\, dx = \int_0^1 g(x) \sum_{|n|\le N} e^{-2\pi i n x}\, dx = \int_0^1 |\sum_{|n|\le N} e^{-2\pi i n x}|\, dx$$

That is, the L^1-norm of the kernel really is the norm of the functional.

The Banach-Steinhaus theorem implies that there is $f \in C^o(\mathbf{R}/\mathbf{Z})$ such that

$$\sup_N |\lambda_N(f)| = +\infty$$

That is, the Fourier series of f does not converge at 0. ///

The result can be strengthened by using Baire's theorem again.

Theorem: There is $f \in C^o(\mathbf{R}/\mathbf{Z})$ such that for a dense G_δ of x in $[0,1]$

$$\sup_N \left| \sum_{|n|\le N} \hat{f}(n)\, e^{2\pi i n x} \right| = \infty$$

In fact, the set of such f is a dense G_δ in $C^o(\mathbf{R}/\mathbf{Z})$.

Proof: Take a dense countable set of points x_j in the interval. Let $\lambda_{j,N}$ be the continuous linear functionals on $C^o(\mathbf{R}/\mathbf{Z})$ defined by

$$\lambda_{j,N}(f) = \sum_{|n|\le N} \hat{f}(n)\, e^{2\pi i n x_j}$$

As in the previous proof, the set E_j of functions f where

$$\sup_N |\lambda_{j,N} f| = +\infty$$

is a dense G_δ, so the intersection

$$E = \cap_j E_j$$

is a dense G_δ. Also

$$s(f,x) = \sup_N |\sum_{|n|\le N} \hat{f}(n)\, e^{2\pi i n x_j}|$$

is the sup of continuous functions, so is lower semicontinuous, and so for any $f \in C^o(\mathbf{R}/\mathbf{Z})$

$$\{x \in [0,1] : s(f,x) = +\infty\}$$

is a G_δ in $[0,1]$. Then for $f \in E$, the corresponding G_δ contains a dense subset (the x_i). ///

Lemma: In a complete metric space X, d (with no isolated points), a dense G_δ is uncountable.

Proof: Suppose that

$$E = \bigcap_n U_n = \{x_1, x_2 \ldots\}$$

is a dense G_δ, where the sets U_n are (necessarily dense) open. Let

$$V_n = U_n - \{x_1, \ldots, x_n\}$$

Then V_n is still dense (by the assumption that there are no isolated points) and open, but $\bigcap_n V_n = \phi$, contradicting the Baire theorem. ///

Theorem: *(Riemann-Lebesgue Lemma)* Let $f \in L^1[0,1]$. Then

$$\hat{f}(n) \to 0$$

Proof: Finite linear combinations of exponentials are dense in $C^o(\mathbf{R}/\mathbf{Z})$ in sup norm, and $C^o[0,1]$ is dense in $L^1[0,1]$. That is, given $f \in L^1$ there is $g \in C^o(\mathbf{R}/\mathbf{Z})$ such that $|f-g|_1 < \varepsilon$ and a finite linear combination h of exponentials such that $|g-h|_\infty < \varepsilon$. Since the sup-norm dominates the L^1-norm, $|f-h|_1 < 2\varepsilon$.

Given such h, for large-enough n the Fourier coefficients are simply 0, by orthogonality of distinct exponentials. Thus,

$$|\hat{f}(n)| = \left| \int_0^1 (f(x) - h(x))\, e^{-2\pi i n x}\, dx \right| \le |f-h|_1 < 2\varepsilon$$

This proves the Riemann-Lebesgue Lemma. ///

Theorem: *(corollary of Baire and Open Mapping)* It is *not* true that every sequence $a_n \to 0$ occurs as the collection of Fourier coefficients of an L^1 function.

Proof: Let c_0 be the collection of sequences $\{a_n : n \in \mathbf{Z}\}$ of complex numbers such that

$$\lim_{n \to \infty} a_n = 0$$

with norm

$$|\{a_n\}| = \sup_n |a_n|$$

This norm makes c_0 a Banach space. Let $T : L^1 \to c_0$ by

$$Tf = \{\hat{f}(n) : n \in \mathbf{Z}\} \in c_0$$

Since

$$|\hat{f}(n)| = \left| \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \right| \le \int_0^1 |f(x)|\, dx = |f|_1$$

it is clear that $|T| \le 1$. In fact, taking $f(x) = 1$ shows that $|T| = 1$.

We check that T is injective: if $\hat{f}(n) = 0$ for every n for some L^1 function f, then for any trigonometric polynomial h we have

$$\int_0^1 f(x)\, h(x)\, dx = 0$$

and by dominated convergence (and density of trigonometric polynomials) the same holds for all *continuous* h. From Lusin's theorem and (again) dominated convergence, the same applies with h being a characteristic function of a measurable set. Thus $f = 0$, proving injectivity of T.

If T were surjective from L^1 to c_0, then the Open Mapping Theorem would assure the existence of $\delta > 0$ such that for every L^1 function f

$$|\hat{f}|_\infty \geq \delta \, |f|_1$$

Let

$$f_N(x) = \sum_{|n| \leq N} e^{-2\pi i n x}$$

On one hand, the sup norm of $\hat{f}_N$ is certainly 1. On the other hand, the computation above shows that the L^1 norm of f_N goes to ∞ as $N \to= \infty$. Thus, there is no such $\delta > 0$. Thus, T cannot be surjective. ///

Remark: For any locally compact Hausdorff space X we can define

$$C_o(X) = C_o^o(X) = \{f \in C^o(X) : \text{for each } \varepsilon > 0 \text{ there is compact } K \text{ such that } |f(x)| < \varepsilon \text{ off } K\}$$

and show that with sup norm this space is *complete*.
