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Basic categorial constructions

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Characterization of an object by *mapping properties* makes proof of *uniqueness* nearly automatic, by standard devices from elementary category theory.

In many situations this means that the appearance of *choice* in construction of the object is an illusion.

Further, in some cases a mapping-property *characterization* is surprisingly elementary and simple by comparison to description by *construction*. Often, an item is already uniquely determined by a subset of its desired properties.

Often, mapping-theoretic descriptions determine *further* properties an object *must* have, without explicit details of its *construction*. Indeed, the common impulse to overtly *construct* the desired object is an over-reaction, as one may not need details of its *internal* structure, but only its *interactions* with other objects.

The issue of *existence* is generally more serious, and only addressed here by means of *example* constructions, rather than by general constructions.

Standard concrete examples are considered: sets, abelian groups, topological spaces, vector spaces.

The real reward for developing this viewpoint comes in consideration of more complicated matters, for which the present discussion is preparation.

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1. *Categories and functors*

A **category** is a batch of things, called the **objects** in the category, and maps between them, called **morphisms**. [1]

The terminology is meant to be suggestive, but it is desirable to be explicit about requirements, making us conscious of things otherwise merely subliminal.

For two objects x, y in the category, $\mathrm{Hom}(x, y)$ is the collection of morphisms from x to y. For $f \in \mathrm{Hom}(x, y)$, the object x is the **source** or **domain**, and y is the **target** or **codomain** or **range**.

[1] The collection of objects in a category is *rarely* small enough to be a *set*! The collection of morphisms between two given objects will be *required* to be a set, as one of the axioms for a category. Such small-versus-large issues are frequently just under the surface when category-language is used, but these will not be a serious danger for us.

We require:

• **Smallness of** $\mathrm{Hom}(x, y)$**:** Each $\mathrm{Hom}(x, y)$ is *small*, in the sense that it is a *set*, meaning that it may safely participate in set-theoretic constructions.

• **Composition:** When the domains and ranges match, morphisms may be composed: for objects x, y, z, for $f \in \mathrm{Hom}(x, y)$ and $g \in \mathrm{Hom}(y, z)$ there is $g \circ f \in \mathrm{Hom}(x, z)$. That is, there is a *set*-map

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \longrightarrow \mathrm{Hom}(x, z)$$

$$f \times g \longrightarrow g \circ f \in \mathrm{Hom}(x, z) \qquad \text{(standard notation despite concommitant perversities)}$$

• **Associativity:** For objects x, y, z, w, for each $f \in \mathrm{Hom}(x, y)$, $g \in \mathrm{Hom}(y, z)$, and $h \in \mathrm{Hom}(z, w)$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

• **Identity morphisms:** For each object x there is an **identity** morphism $\mathrm{id}_x \in \mathrm{Hom}(x, x)$, with properties

$$\begin{cases} \mathrm{id}_x \circ f = f & \text{for all } f \in \mathrm{Hom}(t, x), \quad \text{for all objects } t \\ f \circ \mathrm{id}_x = f & \text{for all } f \in \mathrm{Hom}(x, y), \quad \text{for all objects } y \end{cases}$$

It is reasonable to think of morphisms as probably being functions, so that they are maps *from* one thing *to* another, but the axioms do not dictate this. Thus, no effort is required to define the **opposite category** $\mathcal{C}^{\mathrm{op}}$ of a given category $\mathcal{C}$: the opposite category has the same objects, but

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$$

That is, *reverse the arrows.*

Aspects of *injectivity* and *surjectivity* can be described without talking about *elements* of an object as though it were a *set*. A morphism $f \in \mathrm{Hom}(x, y)$ is a **monomorphism** or **monic** if

$$f \circ g = f \circ h \quad \text{implies} \quad g = h \qquad \text{(for all objects } z\text{, where } g, h \in \mathrm{Hom}(z, x))$$

That is, f can be *cancelled on the left* if and only if it is a monomorphism. The morphism $f \in \mathrm{Hom}(x, y)$ is an **epimorphism** or **epic** if

$$g \circ f = h \circ f \quad \text{implies} \quad g = h \qquad \text{(for all objects } z\text{, with } g, h \in \mathrm{Hom}(y, z))$$

That is, f can be *cancelled on the right* if and only if it is an epimorphism. A morphism is an *isomorphism* if it is both an epimorphism and a monomorphism.

If $f \in \mathrm{Hom}(x, y)$ is monic, then x (with f) is a **subobject** of y. If $f \in \mathrm{Hom}(x, y)$ is epic, then y is a **quotient** (object) of x (by f). In many scenarios, objects are *sets* with additional structure and morphisms are set maps with conditions imposed, and often (but not always) *monic* is equivalent to the set-theoretic *injective* and *epic* is equivalent to the set-theoretic *surjective*.

A **functor** F from one category $\mathcal{C}$ to another category $\mathcal{D}$, written (as if it were a *function*)

$$F : \mathcal{C} \to \mathcal{D}$$

is a pair of maps (each still denoted by F)

$$\begin{cases} F : \mathrm{Objects}(\mathcal{C}) \to \mathrm{Objects}(\mathcal{D}) \\ F : \mathrm{Morphisms}(\mathcal{C}) \to \mathrm{Morphisms}(\mathcal{D}) \end{cases}$$

such that

$$\begin{cases} F(\mathrm{Hom}(x,y)) \subset \mathrm{Hom}(Fx,Fy) & \text{for all objects } x,y \text{ in } \mathcal{C} \\ F(f\circ g) = Ff\circ Fg & \text{for all morphisms } f,g \text{ in } \mathcal{C} \text{ with } f\circ g \text{ defined} \end{cases}$$

A **contravariant functor** $F:\mathcal{C}\to\mathcal{D}$ is a pair of maps (also denoted simply by F)

$$\begin{cases} F : \mathrm{Objects}(\mathcal{C}) \to \mathrm{Objects}(\mathcal{D}) \\ F : \mathrm{Morphisms}(\mathcal{C}) \to \mathrm{Morphisms}(\mathcal{D}) \end{cases}$$

so that

$$\begin{cases} F(\mathrm{Hom}(x,y)) \subset \mathrm{Hom}(Fy,Fx) & \text{for all objects } x,y \text{ in } \mathcal{C} \\ F(f\circ g) = Fg\circ Ff & \text{for all morphisms } f,g \text{ in } \mathcal{C} \text{ with } f\circ g \text{ defined} \end{cases}$$

A contravariant functor reverses arrows and reverses the order of composition of morphisms.

That is, *contravariant* refers to the reversal of arrows. A functor which does *not* reverse arrows is called **covariant**, if this requires emphasis. Otherwise, by default *covariance* is understood. Or, often, the issue of whether a functor is *covariant* or *contravariant* can be discerned easily, so hardly bears discussing.

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2. *Standard (boring) examples*

Virtually everything fits into the idea of category, morphism, and functor. The examples below contain nothing surprising. In all these cases, *monomorphisms* really are *injective* and *epimorphisms* really are *surjective*. We have categories in which

- Objects are *sets* and morphisms are maps from one set to another.
- Objects are *finite sets* and morphisms are maps from one set to another.
- Objects are *topological spaces* and morphisms are *continuous maps*.
- Objects are *groups* and morphisms are *group homomorphisms*.
- Objects are *finite groups* and morphisms are *group homomorphisms*.
- Objects are *abelian groups* and morphisms are *group homomorphisms*.
- Objects are *finite abelian groups* and morphisms are *group homomorphisms*.
- Objects are *complex vectorspaces* and morphisms are *complex-linear maps*.
- Fix a ring R. Objects are *modules* over R and morphisms are R-module homomorphisms.

The simplest functors are those which *forget* some structure, called **forgetful functors**:

- The forgetful functor from (the category of) topological spaces to (the category of) sets, which sends a topological space to the *underlying set*, and sends continuous maps to themselves (forgetting that they were continuous before the topologies on the spaces were forgotten!).

- Fix a ring R. The forgetful functor from (the category of) R-modules to abelian groups, obtained by forgetting the *scalar multiplication* by R, but remembering the *addition* in the modules.

A less trivial (but still straightforward) class of functors consists of *dualizing functors*. For example, there is the *contravariant* functor δ which takes a complex vectorspace V to its dual defined as

$$V^* = \{ \text{ complex-linear maps } \lambda : V \to \mathbf{C}\}$$

This is indeed *contravariant*, because for a linear map $f : V \to W$ and $\lambda \in W^*$ there is the natural

$$\delta(f)(\lambda)(v) = \lambda(f(v))$$

describing the image $f \circ \lambda \in V^*$. That is,

$$\delta(f) : W^* \to V^*$$

It is easy to check by symbol-pushing that composition is respected, in the sense that

$$\delta(f \circ g) = \delta(g) \circ \delta(f)$$

Note that this reversal of order of composition is to be expected.

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3. *Initial and final objects*

The innocuous result here, in suitable circumstances, illustrates the essence of the power of elementary category theory. The benefits will be clear when we construct *categories of diagrams*.

[3.0.1] Proposition: Let $f \in \mathrm{Hom}(x, y)$. If there is $g \in \mathrm{Hom}(y, x)$ so that $f \circ g = \mathrm{id}_y$, then f is an *epimorphism*. If there is $g \in \mathrm{Hom}(y, x)$ so that $g \circ f = \mathrm{id}_x$, then f is a *monomorphism*.

Proof: Suppose that $f \circ g = \mathrm{id}_y$. Let z be any object and suppose that $p, q \in \mathrm{Hom}(y, z)$ are such that

$$p \circ f = q \circ f$$

Then compose with g on the right to obtain

$$p = p \circ \mathrm{id}_y = p \circ (f \circ g) = (p \circ f) \circ g$$

$$= (q \circ f) \circ g = q \circ (f \circ g) = q \circ \mathrm{id}_y = q$$

proving that f is an epimorphism. The proof of the other assertion is nearly identical. ///

An **initial object** (if it exists) in a category $\mathcal{C}$ is an object 0 such that, for all objects x in $\mathcal{C}$ there is *exactly one* element in $\mathrm{Hom}(0, x)$.

A **final object** (if it exists) in $\mathcal{C}$ is an object 1 such that, for all objects x in $\mathcal{C}$ there is *exactly one* element in $\mathrm{Hom}(x, 1)$.

Final and initial objects together are **terminal objects**.

[3.0.2] Proposition: Two initial objects 0 and $0'$ in a category $\mathcal{C}$ are *uniquely isomorphic*. That is, there is a *unique* isomorphism

$$f : 0 \to 0'$$

Likewise, two final objects 1 and $1'$ in a category $\mathcal{C}$ are uniquely isomorphic.

Proof: Let $f : 0 \to 0'$ be the unique morphism from 0 to $0'$, and let $g : 0' \to 0$ be the unique morphism from $0'$ to 0. Then $f \circ g$ is a morphism from $0'$ to itself, and $g \circ f$ is a morphism from 0 to itself. In both cases, since there is just one map from 0 to itself and just one from $0'$ to itself, namely the respective *identity maps*, then it must be that

$$f \circ g = \mathrm{id}_{0'} \qquad g \circ f = \mathrm{id}_0$$

That is, f and g are mutual inverses. Thus, f is both monic and epic, so is an isomorphism. The same applies to g. ///

4. *Categories of diagrams: products, coproducts*

Products and *coproducts* will be introduced in a fashion that will make clear that they are (respectively) final and initial objects in more complicated categories. These more complicated categories, *categories of diagrams*, are built up from configurations of morphisms of simpler underlying categories. Concrete examples are given in the next section.

Objects are described by telling *how they map to or from other objects*, rather than by describing their *internal structure*. These mapping properties are also often called **universal** mapping properties for emphasis.

The first example introduces the general idea. Fix a category $\mathcal{C}$ and an object x in $\mathcal{C}$. Let $\mathcal{C}_x$ be the category whose objects are *morphisms to* x. That is, the objects in the new category are morphisms $f \in \mathrm{Hom}(y,x)$ for objects $y \in \mathcal{C}$. For two such objects

$$f \in \mathrm{Hom}_{\mathcal{C}}(y,x) \qquad g \in \mathrm{Hom}_{\mathcal{C}}(z,x)$$

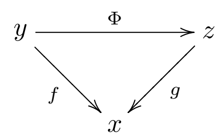
we declare a morphism

$$\Phi \in \mathrm{Hom}_{\mathcal{C}_x}(f,g)$$

to be a morphism $\Phi \in \mathrm{Hom}_{\mathcal{C}}(y,z)$ so that

$$f = g \circ \Phi$$

That is, the morphisms Φ in $\mathcal{C}_x$ from f to g are exactly the morphisms Φ in $\mathcal{C}$ from y to z such that the following diagram *commutes*



meaning that whichever route is taken from y in the upper left to the copy of x at the bottom the same result occurs. Sometimes a category such as this $\mathcal{C}_x$ is called a **comma category**.

More generally, and informally, any category whose objects are some sort of configuration of morphisms from another category will be called a **category of diagrams**.

Next, consider an extension of the previous idea. Fix an index set I and a collection $X = \{x_i : i \in I\}$ of objects x_i of $\mathcal{C}$. Let $\mathcal{C}_X$ be the category whose objects are *collections of morphisms to the* x_i. That is, the objects in the new category are $f_i \in \mathrm{Hom}(y, x_i)$ for objects $y \in \mathcal{C}$, for all indices i. For two such objects $f = \{f_i : i \in I\}$ and $g = \{g_i : i \in I\}$ with

$$f_i \in \mathrm{Hom}_{\mathcal{C}}(y,x_i) \qquad g_i \in \mathrm{Hom}_{\mathcal{C}}(z,x_i)$$

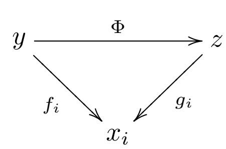
declare a morphism

$$\Phi \in \mathrm{Hom}_{\mathcal{C}_X}(\{f_i\},\{g_i\})$$

to be a morphism $\Phi \in \mathrm{Hom}_{\mathcal{C}}(y,z)$ such that

$$f_i = g_i \circ \Phi \qquad \forall i \in I$$

That is, the morphisms Φ in $\mathcal{C}_X$ from f to g are exactly the morphisms Φ in $\mathcal{C}$ from y to z such that



(for all $i \in I$)

commutes, meaning as above that whichever route is taken from y in the upper left to the copy of x_i at the bottom the same result occurs.

The map Φ is *compatible with* or *respects* the constraints imposed by the maps f_i and g_i. [2]

A *final object* in the previous category $\mathcal{C}_X$ is the definition of **product** of the objects x_i. More precisely, suppose that there is an object y_{final} in $\mathcal{C}$ and morphisms p_i from y_{final} to x_i so that, for any object y in $\mathcal{C}$ and morphisms g_i from y to the x_i there is a unique **induced map**

$$\Phi \in \text{Hom}_{\mathcal{C}}(y, y_{\text{final}})$$

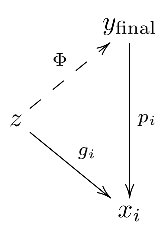
such that for every index i

$$g_i = p_i \circ \Phi$$

Then the object y_{final} *together with the morphisms* p_i is a **product**

$$y_{\text{final}} = \prod_{i \in I} x_i$$

of the objects x_i. The morphisms p_i are the **projections**. [3] Diagrammatically, a dotted arrow is often used to indicate existence. The property of a product y_{final} with projections $p_i : y_{\text{final}} \to x_i$ is expressed as



(for all z and compatible maps $g_i : z \to x_i$)

That is, the dotted arrow indicates that there exists (a unique) Φ making every such triangle commute.

Often, in an abuse of language, we refer to the *object* y_{final} above as the *product*, with implicit reference to the projections. However, it is important to realize that projections must be specified or understood.

Before giving concrete examples of products, *uniqueness* of products can be proven *from general principles.* This illustrates the point that observations about terminal objects which seem trite for mundane categories may have substance for *categories of diagrams.*

[4.0.1] Proposition: Fix a category $\mathcal{C}$ and a family of objects $\{x_i : i \in I\}$ in $\mathcal{C}$. *If* there *is* a product y (with *projections* p_i) then the product is *unique up to unique isomorphism.*

Proof: In this context, a product (if it exists) is simply a final object in a category of diagrams. The simple general result about uniqueness of initial and final objects applies. ///

Reverse the arrows in the discussion of products to obtain an analogous definition of **coproduct** (also called **direct sum** in certain circumstances), as follows.

[2] At a lower level, when maps g_i and f_i are given and a map Φ is found such that $f_i = g_i \circ \Phi$, the map f_i is said to *factor through* g_i, meaning that it can be written as a composite with g_i. It is ambiguous whether $f \circ g_i$ or $g_i \circ f$ is meant.

[3] The terminology *projection* suggests that these maps are epic, and typically they are, but there is no *requirement* or *insinuation* that they be epic.

Fix an index set I and a collection $X = \{x_i : i \in I\}$ of objects x_i of $\mathcal{C}$. Let $\mathcal{C}^X$ be the category whose objects are *collections of morphisms from the* x_i. That is, the objects of the new category are $f_i \in \mathrm{Hom}(x_i, y)$ for objects $y \in \mathcal{C}$, for all indices i. For two such objects $f = \{f_i : i \in I\}$ and $g = \{g_i : i \in I\}$ with

$$f_i \in \mathrm{Hom}_{\mathcal{C}}(x_i, y) \qquad g_i \in \mathrm{Hom}_{\mathcal{C}}(x_i, z)$$

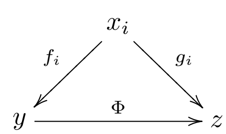
we declare a morphism

$$\Phi \in \mathrm{Hom}_{\mathcal{C}^X}(\{f_i\}, \{g_i\})$$

to be a morphism $\Phi \in \mathrm{Hom}_{\mathcal{C}}(y, z)$ so that

$$\Phi \circ f_i = g_i \qquad \forall i \in I$$

That is, the morphisms Φ in $\mathcal{C}^X$ from f to g are exactly the morphisms Φ in $\mathcal{C}$ from y to z so that *for every index* i the following diagram *commutes*



meaning (as above) that whichever route is taken from x_i at the top to z at the bottom same result occurs. That is, Φ is *compatible with* or *respects* the constraints imposed by the f_i and g_i. [4]

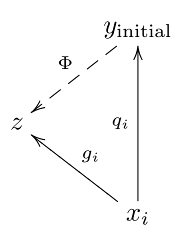
An *initial object* in the previous category $\mathcal{C}^X$ is a **coproduct** of the objects x_i. More precisely, suppose that there is an object y_{initial} in $\mathcal{C}$ and morphisms q_i from x_i to y_{initial} so that, for any object y in $\mathcal{C}$ and morphisms g_i from x_i to y there is a unique **induced map**

$$\Phi \in \mathrm{Hom}_{\mathcal{C}}(y_{\mathrm{initial}}, y)$$

so that for every index i

$$g_i = \Phi \circ q_i$$

The property of a coproduct y_{initial} with morphisms $q_i : x_i \to y_{\mathrm{initial}}$ is expressed diagrammatically as



(for all z and compatible maps $g_i : x_i \to z$)

Then the object y_{initial} *together with the morphisms* q_i is a **coproduct**

$$y_{\mathrm{initial}} = \coprod_{i \in I} x_i$$

of the objects x_i. The morphisms q_i are the **(canonical) inclusions**. [5]

[4] At a lower level, when f_i and g_i are given and Φ is found such that $\Phi \circ f_i = g_i$, the map g_i is said to *factor through* f_i, meaning that g_i can be written as a composite with f_i.

[5] The use of the term *inclusion* suggests that these morphisms are monic, and this is generally so, but is neither required nor implied.

Very often, by abuse of language, we refer to the *object* y above as the *coproduct*, with some implicit reference to the inclusions. It is important to realize, though, that inclusions must be specified, if only tacitly so depending upon context.

[4.0.2] Proposition: Fix a category $\mathcal{C}$ and a family of objects $\{x_i : i \in I\}$ in $\mathcal{C}$. *If* there *exists* a coproduct y (with inclusions q_i) *then* the coproduct is *unique up to unique isomorphism.*

Proof: In this context, a coproduct (if it exists) is an initial object in a category of diagrams. The general result about uniqueness of initial and final objects applies. ///

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5. *Example: sets*

We will examine products and coproducts in several familiar concrete categories. The first case to be considered, with least structure, is *sets and set maps.* That is, we look at the category whose objects are sets, and whose morphisms are arbitrary maps between sets. [6]

First, we will see that *products* in the category of sets are the usual *cartesian products.* Thus, given sets x_i with index i in a set I, let y be the collection of functions [7] φ on I so that $\varphi(i) \in x_i$. Define the i^{th} *projection* p_i to be the function which takes the i^{th} coordinate in the sense that

$$p_i(\varphi) \;=\; \varphi(i)$$

This is the usual description of the *cartesian* product.

Let's verify that the cartesian product has the asserted *mapping properties* making it a *final object* in a suitable *category of diagrams*, so it is a product. Given a collection of maps $f_i : z \to x_i$ from some set z to the x_i, on one hand define

$$\Phi : z \to y$$

by

$$\Phi(\zeta)(i) = f_i(\zeta) \qquad \text{(for } \zeta \in z\text{)}$$

That is, the i^{th} coordinate of $\Phi(\zeta)$ is $f_i(\zeta)$. If we show that there is no *other* map from z to the alleged product y with the the requisite mapping property, then y is a final object as desired.

The mapping requirement

$$p_i \circ \Phi = f_i$$

determines all the coordinates of the image of each $\Phi(\zeta)$ for $\zeta \in z$. This is the required uniqueness. Thus, we have proven that *the usual cartesian product is a product in the category of sets.* Thus, the usual connotation of the notation

$$\text{product of the } x_i = \prod_{i \in I} x_i$$

matches the categorical sense.

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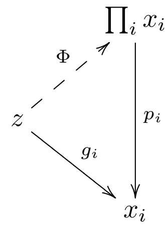
[6] The category of sets is an example of a category whose collection of objects is too large to be a set.

[7] This notion of function is not immediately legitimate as it stands, since functions are required to take values in a fixed common set, while the current functions take values in different sets for different inputs. This can be remedied by declaring the functions to take values in the *disjoint union* of the sets x_i. Once the non-profundity of this remedy is observed, it may seem less urgent to apply it overtly.

Again, the diagrammatical version of the requirements for a product $\prod_i x_i$ is that, for all g_i, there is a unique Φ:



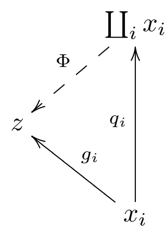
(for all z and compatible maps $g_i : z \to x_i$)

Next, we claim that the *coproduct* y of sets x_i is the *disjoint union:*

$$y \;=\; \coprod_{i\in I} x_i = \bigsqcup_{i\in I} x_i$$

That is, suppressing foundational issues, we must manage to *view* the sets x_i as mutually disjoint.

The diagrammatic requirement is that for all g_i there is a unique compatible Φ, that is,



(for all z and compatible maps $g_i : x_i \to z$)

First, the *inclusions* $q_i : x_i \to y$ are taken to be the obvious inclusions of the x_i into the disjoint union. Given a collection of maps $f_i : x_i \to z$, define the *induced map* $f : y \to z$ by taking $f(\eta) = f_i(\eta)$ for $\eta \in x_i \subset y$. Since each $\eta \in y$ lies inside a unique (copy of) x_i inside y, this is well-defined.

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6. *Example: topological spaces*

The next example of products is in the category of *topological spaces* and *continuous maps.* We claim that products in the category of topological spaces are cartesian products with the *product topology.* We do not have to *make up* a topology to put on the set-product, but simply understand what the categorical product requires. [8] In different words, we want to *discover* the product topology.

The *underlying set* of the product of topological spaces is the same as the product in the category of sets: this is not a coincidence, but does not *follow* from the nature of the set-theoretic product.

A new feature concerning products is *continuity*: the issues of *existence* and *uniqueness* of the induced set-map is already settled. There are two continuity issues. The first does not depend on any other family of maps $f_i : z \to x_i$, being continuity of the *projections*

$$p_i : y \to x_i$$

[8] Recall that the standard product topology on the cartesian product $\prod_{i\in I} x_i$ has a sub-basis consisting of open sets of the form

$$U_{i_o} \times \prod_{i\neq i_o} x_i$$

That is, the sub-basis consists of open sets which are themselves cartesian products of subsets of the x_i, wherein *all but one* of the factors is the whole x_i. One motivation for discussion of this example is that the product topology is surprisingly *coarse* when the index set I is infinite. We want a *reason* for the product topology being what it is.

from the product y to the factors x_i. On the other hand, given a collection of maps $f_i : z \to x_i$ we must prove continuity of the *induced map* $\Phi : z \to y$ through which all these maps *factor* in the sense that

$$f_i = p_i \circ \Phi$$

There is an element of conflict here, because continuity of the projections is a demand for *finer* topology on the product, while continuity of the induced map is a demand for a *coarser* topology on the product.

Fix an index i_o, and let U be an open set in x_{i_o}. The continuity of the projections requires

$$p_{i_o}^{-1}(U) \;=\; U \times \prod_{i \neq i_o} x_i \;=\; \text{open in the product}$$

Thus, arbitrary unions of finite intersections of these sets are required to be open in the product. The projections are continuous if the topology on the set-product is *at least as fine* as the topology with these sets as sub-basis. [9]

From the other side, *induced maps* $\Phi : z \to y$ from families $g_i : z \to x_i$ are required to be continuous. It suffices to require that Φ^{-1} of the (anticipated sub-basis) opens

$$U \times \prod_{i \neq i_o} x_i$$

are open (where U is open in x_{i_o}). These are easy to understand:

$$\Phi^{-1}(U \times \prod_{i \neq i_o} x_i) \;=\; \{\zeta \in z \;:\; f_{i_o}(\zeta) \in U\}$$

since there is *no condition* imposed via the other maps f_i with $i \neq i_o$. Since f_{i_o} is continuous this inverse image is open. Thus, the induced maps *are* open when the topology on the set-product is *no finer than* the topology generated by the sets $U \times \prod_{i \neq i_o} x_i$.

Finally, observe that the *categorically specified* topology on the set-product *is* the usual product topology. This completes the argument that *cartesian products with the product topology are products in the category of topological spaces.*

In particular, talking about the product topology in the present terms exhibits the inevitability of it. It is desirable that the most important feature, the *mapping property*, be the definition, while the particular internals, the *construction*, be artifacts.

Identification of the *coproduct* in familiar terms is easy but not interesting: it is the disjoint union with each piece given its original topology.

---

7. *Example: products of groups*

Now consider products in the category *groups and group homomorphisms.*

We claim that *products* of groups are *cartesian* products of the underlying sets, with a group operation *canonically induced.* That is, there is no *choice* of operation, because it is unequivocally determined by the situation. We do not have to *invent* an operation to hang on the set-product, but only parse the requirements of the group-product.

[9] The product topology is unique if it exists. Thus, *if* there is a product topology, this collection *must* be a sub-basis. But we also want to prove *existence.*

That is, for a collection of groups $\{G_i : i \in I\}$, the product of sets

$$G = \prod_{i\in I} G_i \qquad \text{(with projections } p_i : G \to G_i)$$

inherits a group operation, as follows. The group operations on the G_i are maps

$$\gamma_i : G_i \times G_i \longrightarrow G_i$$

Composing γ_i with $p_i \times p_i$ gives *set* maps

$$\gamma_i \circ \big(p_i \times p_i\big) : G \times G \longrightarrow G_i \qquad \text{(for all } i)$$

The defining property of the set product $G = \prod_i G_i$ is that this situation gives a unique compatible set map

$$\gamma : G \times G \longrightarrow G \qquad \text{(with } p_i \circ \gamma = \gamma_i \circ (p_i \times p_i))$$

In a diagram, this is

$$\begin{array}{ccc} G \times G & \overset{\gamma}{\dashrightarrow} & G \\ {\scriptstyle p_i \times p_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ G_i \times G_i & \xrightarrow{\gamma_i} & G_i \end{array}$$

We claim that γ has the properties of a group operation on G, and that the projections p_i are group homomorphisms.

If γ gives a group operation on G, then the fact that the projections p_i are group homomorphisms is just the compatibility condition on γ: this is

$$p_i\big(\gamma(x, y)\big) = \gamma_i\big(p_i x,\, p_i y\big) \qquad \text{(with } x, y \in G)$$

To check *associativity*, show that all the projections match.

$$p_i\Big(\gamma\big(\gamma(x, y), z\big)\Big) = (p_i \circ \gamma)\big(\gamma(x, y), z\big) = \Big(\gamma_i \circ (p_i \times p_i)\Big)\big(\gamma(x, y), z\big) \qquad \text{(by compatibility of } \gamma)$$

$$= \gamma_i\Big((p_i\times p_i)\big(\gamma(x, y), z\big)\Big) = \gamma_i\Big((p_i\circ\gamma)(x, y),\, p_i(z)\Big) = \gamma_i\Big(\gamma_i(p_i x, p_i y),\, p_i z\Big) \qquad \text{(compatibility again)}$$

The associativity of multiplication γ_i in G_i gives

$$\gamma_i\Big(\gamma_i(p_i x, p_i y),\, p_i z\Big) = \gamma_i\Big(p_i x,\, \gamma_i(p_i y, p_i z)\Big)$$

and then we reverse the previous computation to obtain

$$p_i\Big(\gamma\big(\gamma(x, y), z\big)\Big) = p_i\Big(\gamma\big(x,\, \gamma(y, z)\big)\Big)$$

Thus, we have two set-maps

$$G \times G \times G \longrightarrow G$$

which agree when post-composed with projections, proving that the two maps are identical, proving the associativity of γ.

The *identity* element in the product is specified by considering set-maps f_i of the one-element set $\{1\}$ to all the G_i such that $f_i(1) = e_i$ with e_i the identity in G_i. The set-product produces a unique map $f : \{1\} \to G$ compatible with projections. Let $e = f(1)$, and prove that it has the property of an identity: for $g \in G$,

$$p_i\, \gamma(e, g) = \gamma_i\big(p_i e,\, p_i g\big) = \gamma_i\big(e_i,\, p_i g\big) = p_i g$$

This proves that $\gamma(e,g) = g$. A symmetrical argument proves that $\gamma(g,e) = g$. That is, e has the defining property of the identity element in G.

Existence of inverses: let $g \in G$, and let f_i map a one-element set $\{1\}$ to G_i by by

$$f_i(1) \;=\; p_i(g)^{-1}$$

These maps induce a unique compatible $f : \{1\} \to G$, so

$$p_i\big(\gamma(g, f(1))\big) \;=\; \gamma_i(p_i g,\, p_i f(1)) \;=\; e_i$$

Thus, all the projections are respective identities, so $f(1)$ is a *right* inverse to g. A symmetrical argument proves that $f(1)$ is also a *left* inverse, so is *the* inverse to g in G. This completes the checking that G equipped with γ is a group.

The fact that the unique compatible *set*-map is a *group* homomorphism is similarly inevitable: given group homomorphisms $\varphi_i : H \to G_i$, let $\varphi : H \to G$ be the induced *set* map, and for $x, y \in H$ compute directly

$$p_i\,\varphi(xy) \;=\; \varphi_i\,(xy) \;=\; \varphi_i(x)\cdot\varphi_i(y) \;=\; (p_i\circ\varphi)(x)\cdot(p_i\circ\varphi)(y) \;=\; p_i\big(\varphi(x)\cdot\varphi(y)\big) \qquad (p_i \text{ is a homomorphism})$$

Since the collection of all projections determines the element of the product, this shows that φ is a group homomorphism.

The same-ness of all the above arguments suggests the possibility of further abstracting these arguments, to prove that products of *sets with operations* are the set-products with canonically induced operations.

---

8. *Coproducts of abelian groups*

Coproducts of abelian groups are also called *direct sums*, and it is here that there is divergence from expectations: coproducts of abelian groups are *not* set-coproducts (disjoint unions) of the underlying sets, with a group operation superposed. This is in contrast to the case of topological spaces, where *both* products and coproducts were the set-theoretic versions with additional structure. [10] Thus, to prove that the construction below yields a coproduct, we must prove existence and uniqueness of the induced map, unlike the case of *products*, where the existence of a set-map was already assured.

Further, because coproducts of non-abelian groups require considerable extra effort, we restrict attention to *abelian* groups. [11]

The claim is that the coproduct

$$S \;=\; \coprod_{i\in I} A_i \;=\; \bigoplus_{i\in I} A_i$$

of abelian groups A_i is the *subgroup* of the *product* $\prod_i A_i$

$$\coprod_{i\in I} A_i \;=\; \{a \in \prod_{i\in I} A_i \;:\; p_i(a) = 0 \;\text{ for all but finitely-many } i\;\} \qquad (p_i \text{ the } i^{th} \text{ projection})$$

The *inclusion maps* $q_i : A_i \to S$ are described by giving all the compositions with projections:

$$p_i\Big(q_j(a_j)\Big) \;=\; \begin{cases} a_j & (\text{for } i = j) \\ 0 & (\text{otherwise}) \end{cases}$$

[10] The forgetful functor from abelian groups to sets evidently does not respect coproducts.

[11] While it is true that coproducts are *described* correctly by *reversing the arrows* in the description of *products*, proof of *existence* by construction is an entirely different matter. Arrows and their directions have *meanings* in concrete categories.

Given a family of group homomorphisms $f_i : A_i \to H$, there is a unique *induced map*

$$\Phi \,:\, S \longrightarrow H$$

defined by

$$\Phi(a) \;=\; \sum_{i\in I} f_i\big(p_i(a)\big)$$

where the sum on the right-hand side is inside H. *All but finitely-many of the summands are the identity element in* H, so the sum makes sense without concern for convergence. We have

$$f_i(a_i) \;=\; f_i(a_i) + \sum_{j\neq i} 0 \;=\; \Phi\big(q_i(a_i)\big)$$

so Φ has the required property of an induced map: the maps f_i *factor through it*:

$$f_i = \Phi \circ q_i$$

It is easy to see that Φ is a *group homomorphism.*

A less obvious but critical issue is proof that there is no *other* map $\Psi : S \to H$ with this property. In any case, the difference map

$$(\Phi - \Psi)(a) = \Phi(a) - \Psi(a_i)$$

would have the property that for all indices i

$$(\Phi - \Psi)\circ q_i \;=\; \text{ the zero-map from } A_i \text{ to } H$$

That is, the *kernel* of the group homomorphism $\Phi - \Psi$ from S to H would contain the images $q_i(A_i)$ of all the groups A_i. This kernel is a sub*group*, so is closed under (finite) sums. Since every element in S is a sum of (finitely-many) elements of the form $q_i(a_i)$, the kernel of $\Phi - \Psi$ is the whole group S. This gives the desired uniqueness, finishing the characterization of the induced map.

The above coproduct construction fails for not-necessarily abelian groups. Specifically, in the expression

$$\Phi(a) \;=\; \sum_{i\in I} f_i\big(p_i(a)\big)$$

for the induced map we should no longer denote the group operation by $+$, making the point that *order* of operations now matters. Even with just *two* groups A_1, A_2 and with a not-necessarily abelian group H, this issue arises. That is, a coproduct of *abelian* groups in the category of *not necessarily abelian* groups is not abelian.

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9. *Example: vectorspaces and duality*

In the category of *vectorspaces* over a field k and k-linear maps, we can consider constructions of product and coproduct entirely analogous to the case of abelian groups, and take *duals*. We will show that the dual of a coproduct $\coprod V_i$ is the product $\prod V_i^*$ of the duals V_i^*, but if the index set is infinite the dual of a product is not readily identifiable.

First, *forgetting* the scalar multiplication on a vectorspace produces an abelian group. Optimistically, we imagine that products and coproducts of vectorspaces are the products and coproducts of the underlying abelian groups, with the additional structure of scalar multiplication. Further, we will see that the added structure of scalar multiplication on products is completely determined from the scalar multiplication on the factors in the product.

Given a scalar α, also let α denote scalar multiplication *by* α on vector spaces V_i. Let p_i be the i^{th} projection from the product $V = \prod_i V_i$. The collection of abelian group maps

$$\alpha \circ p_i \; : \; V \longrightarrow V_i$$

determines a unique compatible *abelian group* map $\widetilde{\alpha} : V \to V$. To prove that these maps are genuine scalar multiplications, let β be another scalar. Then for $v \in V$, since p_i is already an abelian group morphism,

$$p_i\big(\widetilde{\alpha}v + \widetilde{\beta}v\big) \;=\; p_i\widetilde{\alpha}v + p_i\widetilde{\beta}v \;=\; \alpha p_i v + \beta p_i v \;=\; (\alpha+\beta)p_i v \;=\; p_i(\widetilde{\alpha+\beta})v$$

This all proves that these induced maps are scalar multiplications that make the projections p_i vector space maps.

A similar argument adds the additional structure to the abelian-group *coproduct* of vector spaces.

We claim that the *dual* of a *coproduct* $\coprod_i V_i$ is the corresponding product of dual vectorspaces. Let $q_i : V_i \to V$ be the inclusions, and $p_i : V \to V_i$ the projections. There is a family of maps

$$V^* \;=\; \mathrm{Hom}(V,k) \longrightarrow \mathrm{Hom}(V_i,k) \;=\; V_i^* \qquad \text{by} \qquad \lambda \longrightarrow \lambda \circ q_i$$

On the other hand, given a collection of $\lambda_i \in V_i^*$, define

$$\lambda(v) \;=\; \sum_{i\in I} \lambda_i(p_i v) \qquad\qquad (\text{for } v \in V)$$

Since only finitely-many of the v_i are non-zero, the sum on the right-hand side has all but finitely-many summands zero. It is easy to check that these two procedures are mutual inverses, giving the isomorphism.

When the index set is *finite*, the product and coproduct are indistinguishable. However, for *infinite* index set the two are very different.

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10. *Limits*

Limits, also called *projective limits* or *inverse limits*, are *final objects* in categories of diagrams, thus giving the most decisive uniqueness proof. Dually, in the next section, *colimits*, also called *inductive limits* or *direct limits*, are *initial objects* in categories of diagrams.

A **partially-ordered set** or **poset** is a set S with a **partial ordering** $\le$. That is, $\le$ is a binary relation on S with the usual properties: for $x, y, z \in S$,

$$\begin{cases} x \le x & \textit{(reflexivity)} \\ x \le y \text{ and } y \le z \text{ implies } x \le z & \textit{(transitivity)} \\ x \le y \text{ and } y \le x \text{ implies } x = y & \textit{(exclusivity)} \end{cases}$$

No assertion is made about comparability of two elements: given $x, y \in S$, it may be that *neither* $x \le y$ *nor* $y \le x$. *When* it is true that for every $x, y \in S$ either $x \le y$ *or* $y \le x$, the ordering is *total*.

A **directed set** I is a poset with the further property that for $i, j \in I$ there is $k \in I$ such that

$$i \le k \quad \text{and} \quad j \le k$$

That is, there is an *upper bound* z of any pair x, y of elements. No assertion is made about *least* upper bounds.

A simple example of a directed set is the set of positive integers with the usual ordering. This may be the most important example, but there are others, as well, and the flexibility of the general definition is useful.

A **projective system** in a category is a collection of objects $\{x_i : i \in I\}$ indexed by a directed set I, with a collection of maps

$$\varphi_{i,j} \;:\; x_i \longrightarrow x_j \qquad \text{(whenever } i > j\text{)}$$

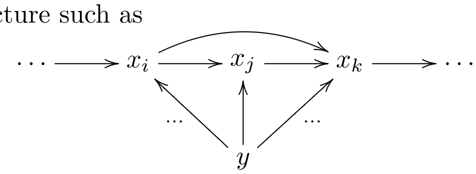
with the **compatibility property**

$$\varphi_{i,k} \;=\; \varphi_{j,k} \circ \varphi_{i,j} \qquad \text{(whenever } i > j > k\text{)}$$

Given a fixed projective system $\{x_i : i \in I\}$ in a category $\mathcal{C}$, with maps $\varphi_{i,j}$ (meeting the *compatibility condition*), we make a *new* category $\mathcal{P}$ whose objects are objects y of $\mathcal{C}$ *together with* maps $f_i : y \to x_i$ satisfying a compatibility condition

$$f_j \;=\; \varphi_{i,j} \circ f_i \qquad \text{(for all } i > j\text{)}$$

In terms of diagrams, often a picture such as

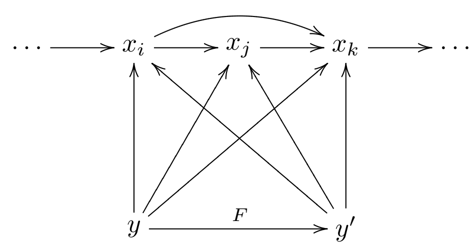


is drawn, and the diagram is said to *commute*, since the same effect is achieved whatever route one takes through the various maps. If possible, often labels in a diagram are suppressed to avoid clutter.

Let y and $\{f_i\}$ be one such object and y' and $\{f'_i\}$ another. The morphisms from y to y' in $\mathcal{P}$ (abusing language in the obvious manner!) are morphisms $F : y \to y'$ such that

$$f'_i \circ F = f_i \qquad \text{(for all indices } i\text{)}$$

In a diagram, this is



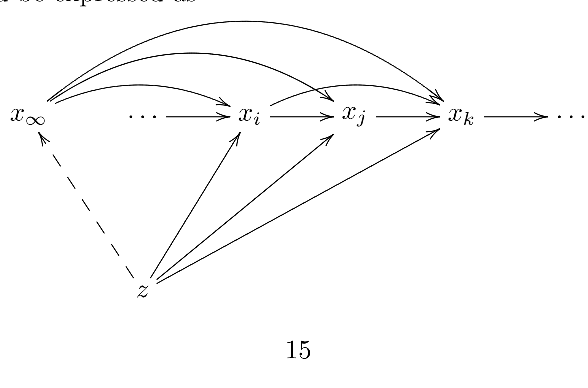
A *final object* in this category $\mathcal{P}$ of diagrams is a **(projective) limit** of the x_i (with reference to the maps $\varphi_{i,j}$ often suppressed).

Thus, a limit of a projective system $\varphi_{i,j} : x_i \to x_j$ is an object x_∞ with maps $\varphi_{\infty,i} : x_\infty \to x_i$ satisfying the *compatibility property* $\varphi_{i,j} \circ \varphi_{\infty,i} = \varphi_{\infty,j}$ for all $i > j$ in the index set.

This batch of stuff has the *universal mapping property* that for any object z and maps $f_i : z \to x_i$ there is a *unique* $\Phi : z \to x_\infty$ so that for all indices

$$f_i = \varphi_{\infty,i} \circ \Phi$$

Diagrammatically, this could be expressed as



where the limit object x_∞ is put on the same line as the x_i to suggest that it is a part of that family of objects. The latter requirements are the *final object* requirement made explicit. The notation for the projective limit is

$$x_\infty = \lim_i x_i$$

with dependence upon the maps typically suppressed. Sometimes a limit is denoted $\underleftarrow{\lim}$, that is, with a backward-pointing arrow under *lim*, probably from the *inverse limit* nomenclature.

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11. *Colimits*

Reversing all the arrows of the previous section defines *colimits*, also called *inductive* or *direct* limits. They are *initial objects* in categories of diagrams.

An **inductive system** in a category is a collection of objects $\{x_i : i \in I\}$ indexed by a directed set I, with maps

$$\varphi_{i,j} : x_i \to x_j \qquad \text{(whenever } i < j\text{)}$$

with the **compatibility**

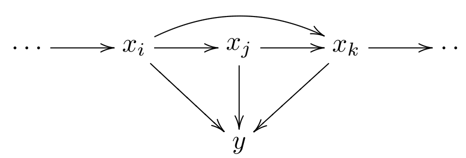
$$\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j} \qquad \text{(for all } i < j < k\text{)}$$

The arrows are in the opposite direction to those in the definiton of a projective system. However, just as the notational symmetry between products and coproducts does not reflect symmetries *in practice*, the notational symmetry between limits and colimits is *not* manifest in practice.

Given an inductive system $\{x_i : i \in I\}$ in a category $\mathcal{C}$, with maps $\varphi_{i,j}$ (meeting the *compatibility condition*), make a *new* category $\mathcal{P}$ whose objects are objects y of $\mathcal{C}$ *with* maps $f_i : x_i \to y$ satisfying a compatibility condition

$$f_i = f_j \circ \varphi_{i,j} \qquad \text{(for all } i < j\text{)}$$

In terms of diagrams, often a picture such as

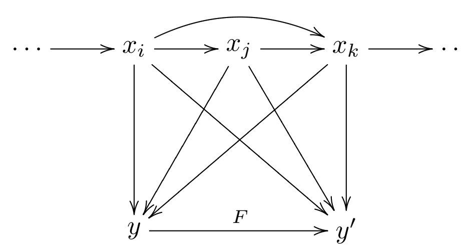


is drawn (suitably labelled), and the diagram is said to *commute*, since the same effect is achieved whatever route one takes through the various maps.

Let y and $\{f_i\}$ be one such object and y' and $\{f'_i\}$ another. The morphisms from y to y' in $\mathcal{P}$ (abusing language in the obvious manner!) are morphisms $F : y \to y'$ so that

$$F \circ f_i = f'_i \qquad \text{(for all indices } i\text{)}$$

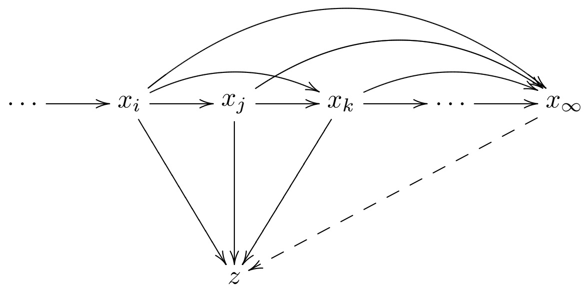
Diagrammatically,



An *initial object* in this category $\mathcal{P}$ of diagrams is a **colimit** or **inductive limit** or **direct limit** of the x_i (with reference to the maps $\varphi_{i,j}$ often suppressed). A colimit of $\varphi_{i,j} : x_i \to x_j$ is an object x_∞ with maps $\varphi_{i,\infty} : x_i \to x_\infty$, and the *compatibility property* $\varphi_{j,\infty} \circ \varphi_{i,j} = \varphi_{i,\infty}$ for all $i < j$ in the index set. Parsing the initial object requirements, the colimit has the *universal mapping property* that for any object z and maps $f_i : x_i \to z$ there is a *unique* $\Phi : x_\infty \to z$ so that for all indices

$$f_i = \Phi \circ \varphi_{i,\infty}$$

Diagrammatically,



The notation for the colimit is

$$x_\infty = \text{colim}_i x_i$$

with dependence upon the maps suppressed. Sometimes this is denoted by $\lim\limits_{\to}$, that is, with a forward-pointing arrow under *lim*.

---

12. *Example: nested intersections of sets*

Limits and colimits have simple illustrations in the category of sets. The issue is not *existence*, but rather that *limits are familiar objects*: *nested intersections* of sets are *(projective) limits*.

In the category of sets, a **nested intersection** is defined to be an intersection $\bigcap_i x_i$ with containments

$$\ldots \subset x_{i+1} \subset x_i \subset x_{i-1} \subset \ldots \subset x_2 \subset x_1$$

It is primarily for convenience that we suppose the sets are indexed by positive integers.

To discuss the possibility of viewing such an intersection as a *(projective) limit* we *name* the inclusions: for $i > j$ we have the inclusion $\varphi_{i,j} : x_i \to x_j$. It is clear that the *compatibility conditions* are met. We claim that the (nested) intersection of the sets x_i is the *limit* of the directed system, with the maps $\varphi_{i,j}$: Let x_∞ be the intersection of the x_i, and let

$$\varphi_{\infty,i} \ : \ x_\infty \longrightarrow x_i$$

be the inclusion maps from the intersection to x_i. We must show that, given any set z and any family of (set) maps

$$f_i \ : \ z \longrightarrow x_i$$

with the compatibility

$$\varphi_{i+1,i} \circ f_{i+1} \ = \ f_i$$

there is a *unique* map

$$\Phi \ : \ z \to \bigcap_i x_i$$

such that for every index i

$$f_i \ = \ \varphi_{\infty,i} \circ \Phi$$

The step-wise compatibility condition $\varphi_{i+1,i} \circ f_{i+1} = f_i$ implies the more general condition $\varphi_{j,i} \circ f_j = f_i$ (for $i < j$) by induction. Since all the sets x_i lie inside x_1 the compatibility conditions assert that for $\zeta \in z$

$$f_i(\zeta) \ = \ \varphi_{i,1} \circ f_i(\zeta) \ = \ f_1(\zeta)$$

That is,

$$f_1 = f_2 = f_3 = f_4 = \ldots$$

Thus, the image $f_1(z)$ of the whole set z lies inside the intersection $x_\infty = \bigcap x_i$ of the x_i. Thus, let

$$\Phi = f_1 : z \longrightarrow x_\infty$$

Again, this is legitimate because we just checked that for all indices i

$$f_i = f_1 = \Phi : z \longrightarrow x_\infty \subset x_i$$

Last, we verify the *uniqueness* of the map Φ. For the same reasons, since $\varphi_{\infty,1}$ is the inclusion $x_\infty \subset x_1$, the only possible map $\Psi : z \longrightarrow x_\infty$ that will satisfy

$$\varphi_{\infty,1} \circ \Psi = f_1$$

is

$$\Phi = f_1 = f_2 = f_3 = \ldots$$

That is, the intersection x_∞ satisfies the universal mapping property required of the limit. ///

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13. *Example: ascending unions of sets*

Now reverse the arrows from the previous discussion of nested *intersections* to talk about ascending *unions* as *colimits*. In the category of sets, a **nested union** is a union $\bigcup_i x_i$ with containments

$$x_1 \subset x_2 \subset \ldots \subset x_{i-1} \subset x_i \subset x_{i+1} \subset \ldots$$

It is primarily for convenience that we choose to index the sets by positive integers. To discuss the possibility of viewing such an union as a *colimit* we *name* the inclusions: for $i < j$ we have the obvious inclusion of $\varphi_{i,j} : x_i \to x_j$. It is clear that the *compatibility conditions* among these maps are met. We claim that the (nested) union of the sets x_i is the *colimit* of the directed system, with the maps $\varphi_{i,j}$:

Let x_∞ be the union of the x_i, and let

$$\varphi_{i,\infty} : x_i \longrightarrow x_\infty$$

be the inclusion maps from the x_i to the union x_∞. We must show that, given any set z and any family of (set) maps

$$f_i : x_i \longrightarrow z$$

with the compatibility

$$f_i = f_{i+1} \circ \varphi_{i,i+1}$$

there is a *unique* map

$$\Phi : \bigcup_i x_i \longrightarrow z$$

so that for every index i

$$f_i = \Phi \circ \varphi_{i,\infty}$$

The step-wise compatibility condition $f_{i+1} \circ \varphi_{i,i+1} = f_i$ implies the more general condition $f_j \circ \varphi_{i,j} = f_i$ (for $i < j$) by induction. The compatibility condition assures that the function $\Phi : x_\infty \to z$ defined as follows is *well-defined*: for ξ in the union, choose an index i large enough so that $\xi \in x_i$, and put

$$\Phi(\xi) = f_i(\xi)$$

The compatibility conditions on the f_i assure that this expression is independent of the index i, for i large enough so that $\xi \in x_i$. Then

$$f_i = \Phi \circ \varphi_{i,\infty}$$

since $\varphi_{i,\infty}$ is inclusion. There is no other map Φ which will satisfy the conditions, since the values of any such Φ on each x_i are completely determined by f_i. That is, the union x_∞ satisfies the universal mapping property required of the colimit. ///

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14. *Cofinal sublimits*

For technical reasons, on many occasions it is helpful to replace a limit or colimit by a limit or colimit over a *smaller* directed set, without altering the limit or colimit. Certainly a smaller collection chosen *haphazardly* ought *not* have the same (co-)limit.

Let I be a directed set, with a subset J. If for every element $i \in I$ there is an element $j \in J$ so that $j \geq i$, then J is **cofinal** in I. When J *is* cofinal in I, it follows immediately that J is also a *directed set*, meaning in particular that for any two elements $j_1, j_2 \in J$ there is $j \in J$ so that $j \geq j_1$ and $j \geq j_2$.

The point is that when a collection of objects indexed by a directed set is shrunken to be indexed by a smaller index set which is nevertheless *cofinal*, the (co-)limit is *the same*. More precisely, it is *uniquely isomorphic* to the original (co-)limit. The proof of this is completely abstract, and is the same for limits or colimits, with the arrows reversed.

[14.0.1] **Proposition:** Let I be a directed set with *cofinal* subset J. Let $\{x_i : i \in I\}$ be an *inductive system* indexed by I, with *compatibility maps* $\varphi_{i,j} : x_i \to x_j$ for $i \leq j$. Let

$$x_\infty = \mathrm{colim}_I\, x_i$$

$$x'_\infty = \mathrm{colim}_J\, x_j$$

be the two colimits, with maps

$$\varphi_{i,\infty} : x_i \longrightarrow x_\infty$$

$$\varphi'_{j,\infty} : x_j \longrightarrow x'_\infty$$

There is a unique isomorphism

$$\tau : x'_\infty \longrightarrow x_\infty$$

so that for $j \in J$

$$\varphi_{j,\infty} = \tau \circ \varphi'_{j,\infty} : x_j \longrightarrow x_\infty$$

[14.0.2] **Proposition:** Let I be a directed set with *cofinal* subset J. Let $\{x_i : i \in I\}$ be a *projective system* indexed by I, with *compatibility maps* $\varphi_{i,j} : x_i \to x_j$ for $i \geq j$. Let

$$x_\infty = \lim_I x_i$$

$$x'_\infty = \lim_J x_j$$

be the two (projective) limits, with maps

$$\varphi_{\infty,i} : x_\infty \longrightarrow x_i$$

$$\varphi'_{\infty,j} : x'_\infty \longrightarrow x_j$$

Then there is a unique isomorphism

$$\tau : x_\infty \longrightarrow x'_\infty$$

so that for $j \in J$

$$\varphi_{\infty,j} \;=\; \varphi'_{\infty,j} \circ \tau \;:\; x_\infty \longrightarrow x_j$$

That is, the two limit *objects* are the same, and the associated *maps* agree in the strongest sense. Most of the following proof is a repetition of the earlier argument about uniqueness of terminal objects.

Proof: We give the proof for *colimits*.

By definition of the colimit x'_∞ of the *smaller* collection, since we have compatible maps $\varphi_{i,\infty} : x_j \to x_\infty$ to the *big* colimit x_∞, there is a *unique* morphism $\tau : x'_\infty \to x_\infty$ through which all the maps $\varphi_{i,\infty}$ factor, in the sense that

$$\varphi_{i,\infty} \;=\; \tau \circ \varphi'_{j,\infty} \;:\; x_j \longrightarrow x_\infty$$

So far, the *cofinality* has not been used. We use it to show that τ is an isomorphism. Given $i \in I$ but not necessarily in J, there is $j \in J$ so that $j \geq i$. We can define

$$\psi_{i,\infty} \;:\; x_i \longrightarrow x'_\infty$$

by

$$\psi_{i,\infty} \;=\; \varphi'_{j,\infty} \circ \varphi_{i,j}$$

By the compatibilities among the maps $\varphi i, j$ we get the same thing no matter which $j \geq i$ is used. In particular, for $i \in J$ simply $\psi_{i,\infty} = \varphi'_{i,\infty}$. Further, the maps $\varphi_{i,j}$ fit together with the $\psi_{i,\infty}$, in the sense that

$$\psi_{i,\infty} \;=\; \psi_{j,\infty} \circ \varphi_{i,j} \qquad\qquad \text{(for all } i \leq j)$$

Thus, by the defining property of the colimit x_∞, there is a *unique* isomorphism $\sigma : x_\infty \to x'_\infty$ through which all the maps $\psi_{i,\infty}$ factor.

Further, consider the collection of maps $\varphi_{i\infty} : x_i \to x_\infty$. By the definition of colimit, there is a *unique* map $t : x_\infty \to x_\infty$ through which the $\varphi_{i,\infty}$ factor in the sense that

$$t \circ \varphi_{i,\infty} \;=\; \varphi_{i,\infty}$$

Since the identity map id_{x_∞} on x_∞ has this property, it must be that $\mathrm{id}_{x_\infty} = t$. But also $\tau \circ \sigma$ has this property, so

$$\tau \circ \sigma \;=\; \mathrm{id}_{x_\infty}$$

Similarly,

$$\sigma \circ \tau \;=\; \mathrm{id}_{x'_\infty}$$

Therefore, σ and τ are mutual inverses, so each is an isomorphism. ///