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# *Vector-Valued Integrals*

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We describe a useful class of topological vectorspaces [1] $V$ so that *continuous compactly-supported* $V$-valued functions have *integrals* with respect to finite Borel measures. Rather than *constructing* integrals as limits following [Bochner 1935], [Birkhoff 1935], *et alia*, we use the [Gelfand 1936]-[Pettis 1938] *characterization* of integrals, which has good functorial properties and gives a forceful categorical reason for uniqueness. The issue is *existence*.

A convenient property of topological vectorspaces guaranteeing existence of Gelfand-Pettis integrals is *quasi-completeness*, discussed below. Hilbert, Banach, Fréchet, and LF spaces fall in this class, as do their weak-star duals, and other spaces of mappings such as the *strong operator topology* on mappings between Hilbert spaces, in addition to the *uniform* operator topology.

A compelling application of this integration theory is to holomorphic vector-valued functions, with well-known application to the resolvents of operators on Hilbert and Banach spaces, as in [Dunford 1938] and [Taylor 1938]. In these sources Liouville's theorem on bounded entire $\mathbb{C}$-valued functions is invoked to prove that a bounded operator on a Hilbert or Banach spaces has non-empty spectrum. A more sophisticated application is to meromorphically-continued Eisenstein series.

Another sort of application of holomorphic and meromorphic vector-valued functions is to *generalized functions*, as in [Gelfand-Shilov 1964], studying *holomorphically parametrized families* of distributions. Many distributions which are not classical functions appear naturally as residues or analytic continuations of families of classical functions with a complex parameter.

A good theory of integration allows a natural treatment of convolutions of distributions with test functions, and related operations.

Two basic auxiliary results are proven about bounded sets in topological vector spaces, the first attributed in [Horvath 1966] to Dieudonné-Schwartz. This result also appears in [Bourbaki 1987], III.5. The second is preparation to invoke a form of the Banach-Steinhaus theorem in situations where the Baire category argument is not directly applicable.

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## 1. *Gelfand-Pettis integrals*

Let $V$ be a complex topological vectorspace. Let $f$ be a measurable $V$-valued function on a measure space $X$. A **Gelfand-Pettis integral** of $f$ would be a vector $I_f \in V$ so that,

$$\lambda(I_f) = \int_X \lambda \circ f \qquad \text{(for all } \lambda \in V^*)$$

[1] All vectorspaces here are *complex*, or possibly *real*, as opposed to $p$-adic or other possibilities.

Assuming that it exists and is unique, this vector $I_f$ would be denoted by

$$I_f \;=\; \int_X f$$

By contrast to *construction* of integrals as limits of finite sums, this definition gives a *property* that no reasonable notion of integral would lack, without asking how the property comes to be. Since this property is an irreducible minimum, this definition of integral is called a **weak integral**.

*Uniqueness* of the integral is immediate when $V^*$ *separates points*, as for *locally convex* spaces by Hahn-Banach. Similarly, *linearity* of $f \to I_f$ follows when $V^*$ separates points. Thus, the issue is *existence*. [2]

The functions we integrate are relatively nice: compactly-supported and continuous, on measure spaces with *finite, positive, Borel* measures. In this situation, all the $\mathbb{C}$-valued integrals

$$\int_X \lambda \circ f$$

exist for elementary reasons, being integrals of compactly-supported $\mathbb{C}$-valued continuous functions on a compact set with respect to a finite Borel measure.

The technical condition on the topological vectorspace $V$ is that *the convex hull of a compact set must have compact closure*. A more intuitive property which *implies* this property is **quasi-completeness** (and local convexity), meaning that *bounded* Cauchy *nets* converge. [3] In all applications, when the compactness of closures of convex hulls of compacts holds, it seems that the space is quasi-complete. Thus, while *a priori* the condition of quasi-completeness is stronger than the compactness condition, no example of a strict comparison seems immediate.

The hypotheses of the following theorem will be verified for Fréchet spaces and their weak star duals, for example. In addition to Hilbert and Banach spaces, this includes *Schwartz functions* and *tempered distributions*.

Further, the hypothesis will include *LF-spaces* and their weak $*$-duals. By definition, an LF space is a countable ascending union of Fréchet spaces so that each Fréchet subspace is closed in the next, with the finest possible topology on the union. These are *strict inductive limits* or *strict colimits* of Fréchet spaces. This includes spaces of *test functions*. The weak $*$-duals of the LF spaces are also quasi-complete, so we can integrate *distribution-valued* functions.

In addition to the *uniform* operator topology on continuous linear maps from one Hilbert space or Banach space to another, quasi-completeness holds for the *strong* and *weak* operator topologies. [4]

[1.0.1] Theorem: Let $X$ be a locally compact Hausdorff topological space with a *finite*, positive, Borel measure. Let $V$ be a locally convex topological vectorspace in which the *closure of the convex hull of a*

[2] Lest there be any doubt, we do require that the integral of a $V$-valued function be a vector in the space $V$ itself, rather than in a larger space containing $V$, such as a double dual $V^{**}$, for example. Some alternative discussions of integration do allow integrals to exist in larger spaces.

[3] In topological vectorspaces which may not have countable local bases, quasi-completeness is more relevant than 'plain' completeness. For example, the weak $*$-dual of an infinite-dimensional Hilbert space is *never* complete, but is always quasi-complete. This example is non-trivial, but helps illustrate the appropriateness of quasi-completeness.

[4] The strong and weak operator topologies, as opposed to the uniform norm topology, are the topologies on operators found useful in representation theory.

*compact set is compact.* It suffices that $V$ be *quasi-complete*. Then *continuous compactly-supported* $V$-valued functions $f$ on $X$ have Gelfand-Pettis integrals. Further,

$$\int_X f \;\in\; \mathrm{meas}\,(X)\cdot\Big(\text{closure of convex hull of } f(X)\Big) \qquad \textit{(Proof later.)}$$

[1.0.2] Remark: The conclusion that the integral of $f$ lies in the closure of a convex hull, is a substitute for the estimate of a $\mathbb{C}$-valued integral by the integral of its absolute value.

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# 2. *Coproducts, colimits of topological vectorspaces*

The intuitive idea of a topological vectorspace being an *ascending union* of subspaces is necessary, but must be made precise. Spaces of *test functions* on $\mathbb{R}^n$ are obvious prototypes, being ascending unions of Fréchet spaces without being Fréchet spaces themselves. [5]

[2.1] A categorical viewpoint *Nested intersections* are examples of *limits*, and limits are *closed subobjects* of the corresponding *products*, while *ascending unions* are examples of *colimits*, and colimits are *quotients* (by closed subobjects) of the corresponding *coproducts*. [6] Thus, proof of existence of *products* gets us half-way to proof of existence of *limits*, and proof of existence of *coproducts* gets us half-way to existence of *colimits*.

In more detail: locally convex *products* of locally convex topological vector spaces are proved to *exist* by constructing them as products of topological spaces, with the product topology, with scalar multiplication and vector addition *induced*.

Given the existence of locally convex *products*, a *limit* of topological vector spaces $X_\alpha$ with compatibility maps $p^\alpha_\beta : X_\alpha \to X_\beta$ is proven to exist by constructing it as the (closed) subobject $Y$ of the product $X$ of the $X_\alpha$ consisting of $x \in X$ meeting the compatibility conditions

$$p_\beta(x) \;=\; (p^\alpha_\beta \circ p_\alpha)(x) \qquad \text{(for all } \alpha < \beta\text{, where } p_\ell \text{ is the } \textit{projection } X \to X_\ell)$$

Since local convexity is preserved by these constructions, these constructions do take place inside the category of *locally convex* topological vector spaces.

[2.2] Coproducts and colimits *Locally convex* coproducts $X$ of topological vector spaces $X_\alpha$ are coproducts of the vector spaces $X_\alpha$ with the *diamond topology*, described as follows. [7] For a collection $U_\alpha$ of convex neighborhoods of $0$ in the $X_\alpha$, let

$$U \;=\; \text{convex hull in } X \text{ of the union of } j_\alpha(U_\alpha) \qquad (\text{with } j_\alpha : X_\alpha \to X \text{ the } \alpha^{th} \text{ canonical map})$$

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[5] A countable ascending union of complete metric spaces, with each a proper closed subspace of the next, *cannot* be complete metric, because it is *presented* as a countable union of nowhere-dense closed subsets, contradicting the conclusion of the Baire Category Theorem.

[6] Categorical limits are often called *projective* limits, and *colimits* are often called *inductive* or *direct* limits. For our purposes, the terms *limit* and *colimit* suffice.

[7] The product topology of locally convex topological vector spaces is unavoidably locally convex, whether the product is in the category of locally convex topological vector spaces or in the larger category of not-necessarily-locally-convex topological vector spaces. However, coproducts behave differently: the locally convex coproduct of *uncountably many* locally convex spaces is *not* a coproduct in the larger category of not-necessarily-locally-convex spaces. This already occurs with an uncountable coproduct of *lines*.

The diamond topology has local basis at 0 consisting of such $U$. Thus, it is locally convex by construction. (The closedness of points follows from the corresponding property of the $X_\alpha$.) Thus, existence of a *locallly convex* coproduct (of locally convex spaces) is assured by the construction. The *locally convex* colimit of the $X_\alpha$ with compatibility maps $j^\alpha_\beta : X_\alpha \to X_\beta$ is the *quotient* of the locally convex coproduct $X$ of the $X_\alpha$ by the *closure* of the subobject $Z$ spanned by vectors

$$j_\alpha(x_\alpha) - (j_\beta \circ j^\alpha_\beta)(x_\alpha) \qquad\qquad (\text{for all } \alpha < \beta \text{ and } x_\alpha \in X_\alpha)$$

Annihilation of these differences in the quotient forces the desired compatibility relations. Obviously, quotients of locally convex spaces are locally convex.

**[2.3] Strict colimits** Let $V$ be a locally convex topological vectorspace with subspaces

$$V_o \subset V_1 \subset V_2 \subset V_3 \subset \ldots \qquad\qquad (\text{proper containments, } V_i \textit{ closed} \text{ in } V_{i+1})$$

and

$$V \;=\; \bigcup_i V_i \;=\; \mathrm{colim}_i V_i$$

Such $V$ is a **strict (locally convex) colimit** or **strict (locally convex) inductive limit** of the $V_i$, *strict* because $V_i$ is closed in $V_{i+1}$. The fact that the collection of subspaces is *countable* and *well-ordered* is also a special feature. We noted above that locally convex colimits *exist.*

Because of the strictness, we can show that each $V_i$ *injects* to $V$, as follows. By Hahn-Banach and induction, the identity map $T_i : V_i \to V_i$ extends compatibly to $T_\beta : V_\ell \to V_i$ for all $\ell$, then yielding a unique compatible $T : V \to V_i$. The compatibility

$$T \circ j_i \;=\; T_i \;=\; \text{identity on } V_i \qquad\qquad (\text{with } j_i : V_i \to V)$$

implies that $j_i$ is injective. Similarly, each $V_i$ is *closed* in $V$, as follows. For $v \in V$ but $v \notin V_i$, there is $\ell > i$ such that $v \in V_\ell$, and the quotient map

$$q_\ell \;:\; V_\ell \;\longrightarrow\; V_\ell/V_i$$

is a map of topological vector spaces (since $V_i$ is closed in $V_\ell$) and does not map $v$ to 0. Using Hahn-Banach and well-ordering, $q_\ell$ extends to compatible maps $q_k : V_k \to V_k/V_i$ for all $k > i$, and, thus, to a unique compatible map $q : V \to V_\ell/V_i$. The compatibility assures that $q(v) \neq 0$. That is, $v$ could not be in the closure of $V_i$, or else it could not be mapped to something non-zero by a continuous topological vector space map that maps $V_i$ to $\{0\}$.

**[2.3.1] Proposition:** Assume that $V$ is a *locally convex strict colimit* of a countable well-ordered collection of closed subspaces $V_i$. A subset $B$ of $V$ is *bounded* if and only it lies inside some subspace $V_i$ and is a bounded subset of $V_i$.

*Proof:* Suppose $B$ does *not* lie in any $V_i$. Then there is a sequence $i_1, i_2, \ldots$ of positive integers and $x_{i_\ell}$ in $V_{i_\ell} \cap B$ with $x_{i_\ell}$ *not* lying in $V_{i_\ell - 1}$. Using the simple nature of the indexing set and the simple inter-relationships of the subspaces $V_i$,

$$V = \bigcup_j V_{i_\ell}$$

In particular, without loss of generality, we may suppose that $i_\ell = \ell$.

By the Hahn-Banach theorem and induction, using the closedness of $V_{i-1}$ in $V_i$, there are continuous linear functionals $\lambda_i$ on $V_i$'s such that $\lambda_i(x_i) = i$ and the restriction of $\lambda_i$ to $V_{i-1}$ is $\lambda_{i-1}$, for example. Since $V$ is the colimit of the $V_i$, this collection of functionals exactly describes a unique compatible continuous linear functional $\lambda$ on $V$.

But $\lambda(B)$ is *bounded* since $B$ is bounded and $\lambda$ is continuous, which precludes the possibility that $\lambda$ takes on all positive integer values at the points $x_i$ of $B$. Thus, it could *not* have been that $B$ failed to lie inside some single $V_i$. The strictness of the colimit implies that $B$ is bounded as a subset of $V_i$, proving one direction of the equivalence. The other direction of the equivalence is less interesting. ///

**[2.4] Equicontinuity on strict colimits** Recall that a set $E$ of continuous linear maps from one topological vectorspace $X$ to another topological vectorspace $Y$ is **equicontinuous** when, for every neighborhood $U$ of 0 in $Y$, there is a neighborhood $N$ of 0 in $X$ so that $T(N) \subset U$ for every $T \in E$.

**[2.4.1] Proposition:** Let $V$ be a strict colimit of a well-ordered countable collection of locally convex closed subspaces $V_i$. Let $Y$ be a locally convex topological vectorspace. Let $E$ be a set of continuous linear maps from $V$ to $Y$. Then $E$ is *equicontinuous* if and only if for each index $i$ the collection of continuous linear maps $\{T|_{V_i} : T \in E\}$ is equicontinuous.

*Proof:* Given a neighborhood $U$ of 0 in $Y$, shrink $U$ if necessary so that $U$ is convex and balanced. For each index $i$, let $N_i$ be a convex, balanced neighborhood of 0 in $V_i$ so that $T(N_i) \subset U$ for all $T \in E$. Let $N$ be the convex hull of the union of the $N_i$. By the convexity of $N$, still $T(N) \subset U$ for all $T \in E$. By the construction of the diamond topology, $N$ is an open neighborhood of 0 in the coproduct, hence in the colimit, giving the equicontinuity of $E$. The other direction of the implication is easy. ///

# 3. *Ubiquity of quasi-complete spaces*

This section shows that quasi-completeness is preserved by various important constructions on topological vector spaces, so that most topological vector spaces of interest are quasi-complete. Thus, the theorem on Gelfand-Pettis integrals applies to all these.

Again, a topological vectorspace is **quasi-complete** if every *bounded* Cauchy net is *convergent*. Note that without the assumption that there is a countable local basis (at 0) it is necessary to consider *nets* rather than simply *sequences*. Since many spaces of interest occurring as weak star duals certainly fail to be locally countable, this distinction is not frivolous.

Certainly Fréchet spaces (hence Hilbert and Banach spaces) are quasi-complete, since in the case of a metric space quasi-completeness and ordinary completeness are identical. [8]

It is clear that *closed subspaces* of quasi-complete spaces are quasi-complete. Products (with product topology) and finite sums of quasi-complete spaces are quasi-complete.

**[3.0.1] Proposition:** A strict colimit of a countable collection of closed quasi-complete spaces is quasi-complete.

*Proof:* We saw that bounded subsets of such colimits are exactly the bounded subsets of the limitands. Thus, bounded Cauchy nets in the colimit must be bounded Cauchy nets in one of the closed subspaces. Each of these is assumed quasi-complete, so the colimit is quasi-complete. ///

As a consequence of the proposition, spaces of test functions are quasi-complete, since they are such colimits of the Fréchet spaces of spaces of test functions with prescribed compact support.

Let $\mathrm{Hom}^o(X,Y)$ be the space of continuous linear functions from a topological vectorspace $X$ to another topological vectorspace $Y$. Give $\mathrm{Hom}^o(X,Y)$ the topology induced by the seminorms $p_{x,U}$ where $x \in X$ and $U$ is a convex, balanced neighborhood of 0 in $Y$, defined by

$$p_{x,U}(T) \;=\; \inf\{t > 0 : Tx \in tU\} \qquad \text{(for } T \in \mathrm{Hom}^o(X,Y))$$

[8] That sequential completeness implies completeness for metric spaces is an interesting foundational exercise.

**[3.0.2] Remark:** In the case that $X$ and $Y$ are Hilbert spaces, this construction gives the *strong operator topology* on $\mathrm{Hom}^o(X,Y)$. Replacing the topology on the Hilbert space $Y$ by its *weak* topology, the construction gives $\mathrm{Hom}^o(X,Y)$ the *weak operator topology.* That the collection of continuous linear operators is the same in both cases is a consequence of the Banach-Steinhaus theorem, which also plays a role in the following result.

**[3.0.3] Theorem:** When $X$ is a Fréchet space or LF space, and when $Y$ is quasi-complete, the space $\mathrm{Hom}^o(X,Y)$, with the topology induced by the seminorms $p_{x,U}$ where $x \in X$ and $U$ is a convex, balanced neighborhood of 0 in $Y$, is *quasi-complete.*

**[3.0.4] Remark:** In fact, the starkest hypothesis on $\mathrm{Hom}^o(X,Y)$ is simply that it support the conclusion of the Banach-Steinhaus theorem. That is, a subset $E$ of $\mathrm{Hom}^o(X,Y)$ so that the set of all images

$$Ex = \{Tx : T \in E\}$$

is bounded (in $Y$) for all $x \in X$ is necessarily *equicontinuous.* When $X$ is a Fréchet space, this is true (by the usual Banach-Steinhaus theorem) for *any* $Y$. Further, by the result above on bounded subsets of special sorts of colimits, we see that the same conclusion holds for $X$ such a colimit.

*Proof:* Let $E = \{T_i : i \in I\}$ be a bounded Cauchy net in $\mathrm{Hom}^o(X,Y)$, where $I$ is a directed set. Of course, attempt to define the limit of the net by

$$Tx \;=\; \lim_i T_i x$$

For $x \in X$ the evaluation map $S \to Sx$ from $\mathrm{Hom}^o(X,Y)$ to $Y$ is continuous. In fact, the topology on $\mathrm{Hom}^o(X,Y)$ is the coarsest with this property. Therefore, by the quasi-completeness of $Y$, for each fixed $x \in X$ the net $T_i x$ in $Y$ is bounded and Cauchy, so converges to an element of $Y$ suggestively denoted $Tx$.

To prove *linearity* of $T$, fix $x_1, x_2$ in $X$, $a, b \in \mathbb{C}$ and fix a neighborhood $U_o$ of 0 in $Y$. Since $T$ is in the closure of $E$, for any open neighborhood $N$ of 0 in $\mathrm{Hom}^o(X,Y)$, there exists

$$T_i \;\in\; E \cap (T + N)$$

In particular, for any neighborhood $U$ of 0 in $Y$, take

$$N = \{S \in \mathrm{Hom}^o(X,Y) : S(ax_1 + bx_2) \in U,\; S(x_1) \in U,\; S(x_2) \in U\}$$

Then

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2)$$
$$=\; (T(ax_1 + bx_2) - aT(x_1) - bT(x_2)) - (T_i(ax_1 + bx_2) - aT_i(x_1) - bT_i(x_2))$$

since $T_i$ is linear. The latter expression is

$$T(ax_1 + bx_2) - (ax_1 + bx_2) + a(T(x_1) - T_i(x_1) + b(T(x_2) - T_i(x_2)$$

$$\in\; U + aU + bU$$

By choosing $U$ small enough so that

$$U + aU + bU \;\subset\; U_o$$

we find that

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;\in\; U_o$$

Since this is true for every neighborhood $U_o$ of 0 in $Y$,

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;=\; 0$$

which proves linearity.

To prove *continuity* of the limit operator $T$, we must first be sure that $E$ is *equicontinuous*. For each $x \in X_j$, the set $\{T_i x : i \in I\}$ is bounded in $Y$, so by Banach-Steinhaus $\{T_i : i \in I\}$ is an equicontinuous set of linear maps from $X_i$ to $Y$. (Each $X_i$ is Fréchet.) From the result of the previous section on equicontinuous subsets of LF spaces, $E$ itself is equicontinuous.

Fix a neighborhood $U$ of 0 in $Y$. Invoking the equicontinuity of $E$, let $N$ be a small enough neighborhood of 0 in $X$ so that $T(N) \subset U$ for all $T \in E$. Let $x \in N$. Choose an index $i$ sufficiently large so that $Tx - T_i x \in U$, vis the definition of the topology on $\mathrm{Hom}^o(X,Y)$. Then

$$Tx \in U + T_i x \subset U + U$$

The usual rewriting, replacing $U$ by $U'$ such that $U' + U' \subset U$, shows that $T$ is continuous. ///

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# 4. *Totally bounded sets in topological vectorspaces*

The point of this section is the last corollary, that *convex hulls of compact sets in Fréchet spaces have compact closures*. This is the key point for existence of Gelfand-Pettis integrals.

In preparation, we review the relatively elementary notion of *totally bounded subset* of a metric space, as well as the subtler notion of *totally bounded subset* of a topological vectorspace.

A subset $E$ of a *complete metric space* $X$ is **totally bounded** if, for every $\varepsilon > 0$ there is a covering of $E$ by *finitely-many* open balls of radius $\varepsilon$. The property of *total boundedness* in a metric space is generally stronger than mere *boundedness*. It is immediate that any subset of a totally bounded set is totally bounded.

[4.0.1] Proposition: A subset of a complete metric space has compact closure if and only if it is *totally bounded*.

*Proof:* Certainly if a set has compact closure then it admits a finite covering by open balls of arbitrarily small (positive) radius.

On the other hand, suppose that a set $E$ is totally bounded in a complete metric space $X$. To show that $E$ has compact closure it suffices to show that any sequence $\{x_i\}$ in $E$ has a Cauchy subsequence.

We choose such a subsequence as follows. Cover $E$ by finitely-many open balls of radius 1. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball $B_1$, and let $i_1$ be the smallest index so that $x_{i_1}$ lies in this ball.

The set $E \cap B_1$ is still totally bounded (and contains infinitely-many elements from the sequence). Cover it by finitely-many open balls of radius 1/2, and choose a ball $B_2$ with infinitely-many elements of the sequence lying in $E \cap B_1 \cap B_2$. Choose the index $i_2$ to be the smallest one so that both $i_2 > i_1$ and so that $x_{i_2}$ lies inside $E \cap B_1 \cap B_2$.

Proceeding inductively, suppose that indices $i_1 < \ldots < i_n$ have been chosen, and balls $B_i$ of radius $1/i$, so that

$$x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i$$

Then cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, call it $B_{n+1}$, containing infinitely-many elements of the sequence. Let $i_{n+1}$ be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \;\in\; E \cap B_1 \cap \ldots \cap B_{n+1}$$

Then for $m < n$ we have

$$d(x_{i_m}, x_{i_n}) \;\leq\; \frac{1}{m}$$

so this subsequence is Cauchy. ///

In a *topological vectorspace* $V$, a subset $E$ is **totally bounded** if, for every neighborhood $U$ of 0 there is a finite subset $F$ of $V$ so that

$$E \subset F + U$$

Here the notation $F + U$ means, as usual,

$$F + U = \bigcup_{v \in F} v + U = \{v + u : v \in F, u \in U\}$$

**[4.0.2] Remark:** In a topological vectorspace whose topology is given by a *translation-invariant* metric, a subset is *totally bounded* in this topological vectorspace sense if and only if it is totally bounded in the metric space sense, from the definitions.

**[4.0.3] Lemma:** In a topological vectorspace the convex hull of a *finite* set is *compact*.

*Proof:* Let the finite set be $F = \{x_1, \ldots, x_n\}$. Let $\sigma$ be the compact set

$$\sigma = \{(c_1, \ldots, c_n) \in \mathbb{R}^n : \sum_i c_i = 1,\ 0 \le c_i \le 1, \text{ for all } i\} \subset \mathbb{R}^n$$

Then the convex hull of $F$ is the continuous image of $\sigma$ under the map

$$(c_1, \ldots, c_n) \to \sum_i c_i x_i$$

so is compact. ///

**[4.0.4] Proposition:** A totally bounded subset $E$ of a *locally convex* topological vectorspace $V$ has totally bounded *convex hull*.

*Proof:* Let $U$ be a neighborhood of 0 in $V$. Let $U_1$ be a *convex* neighborhood of 0 so that $U_1 + U_1 \subset U$. Then for some finite subset $F$ we have $E \subset F + U_1$, by the total boundedness. Let $K$ be the convex hull of $F$, which by the previous result is *compact*. Then $E \subset K + U_1$, and the latter set is *convex*, as observed earlier. Therefore, the convex hull $H$ of $E$ lies inside $K + U_1$. Since $K$ is compact, it is totally bounded, so can be covered by a finite union $\Phi + U_1$ of translates of $U_1$. Thus, since $U_1 + U_1 \subset U$,

$$H \subset (\Phi + U_1) + U_1 \subset \Phi + U$$

Thus, $H$ lies inside this finite union of translates of $U$. This holds for any open $U$ containing 0, so $H$ is totally bounded. ///

**[4.0.5] Corollary:** In a Fréchet space, the closure of the convex hull of a compact set is compact.

*Proof:* A compact set in a Fréchet space (or in any complete metric space) is totally bounded, as recalled above. By the previous result, the convex hull of a totally bounded set in a Fréchet space (or in any locally convex space) is totally bounded. Thus, this convex hull has compact closure, since totally bounded sets in complete metric spaces have compact closure. ///

# 5. *Quasi-completeness and convex hulls of compacts*

Again, a topological vectorspace $X$ is **quasi-complete** if every *bounded* Cauchy *net* converges.

The following proof borrows an idea from the proof of the Banach-Alaoglu theorem. It reduces the general case to the case of Fréchet spaces, treated in the previous section.

[5.0.1] Proposition: In a quasi-complete locally convex topological vectorspace $X$, the closure $C$ of the convex hull $H$ of a compact set $K$ is *compact*.

*Proof:* Since $X$ is locally convex, by the Hahn-Banach theorem its topology is given by a collection of seminorms $v$. For each seminorm $v$, let $X_v$ be the completion of the quotient

$$X/\{x \in X : v(x) = 0\}$$

with respect to the *metric* that $v$ induces on the latter quotient. Thus, $X_v$ is a Fréchet space. (Indeed, the latter quotient is the largest quotient of $X$ on which $v$ induces a *metric* rather than merely a *pseudometric*. And, in fact, $X_v$ is Banach, but we don't use this.) Consider

$$Z = \prod_v X_v \qquad\qquad \text{(with product topology)}$$

with the natural injection $j : X \to Z$, and with projection $p_v$ to the $v^{\text{th}}$ factor.

By construction, and by definition of the topology given by the seminorms, $j$ is a homeomorphism to its image. That is, $X$ is homeomorphic to the subset $jX$ of $Z$, given the subspace topology.

The image $p_v jK$ is compact, being a continuous image of a compact subset of $X$. Since $X_v$ is Fréchet, the convex hull $H_v$ of $p_v jK$ has compact closure $C_v$. The convex hull $jH$ of $jK$ is contained in the product $\prod_v H_v$ of the convex hulls $H_v$ of the projections $p_v jK$. By Tychonoff's theorem, the product $\prod_v C_v$ is *compact*.

Since $jC$ is contained in the compact set $\prod_v C_v$, to prove that the closure $jC$ of $jH$ in $jX$ is compact, it suffices to prove that $jC$ is closed in $Z$. Since $jC$ is a subset of the compact set $\prod_v C_v$, it is totally bounded and so is certainly bounded (in $Z$, hence in $X \approx jX$). By the quasi-completeness, any Cauchy net in $jC$ converges to a point in $jC$. Since any point in the closure of $jC$ in $Z$ has a Cauchy net in $jC$ converging to it, $jC$ is closed in $Z$. This finishes the proof that quasi-completeness implies the compactness of closures of compact hulls of compacta. ///

# 6. *Existence of integrals*

Now we will prove existence of integrals: assume that in the topological vectorspace $V$ convex hulls of compacta have compact closures, and prove the *existence* of Gelfand-Pettis integrals $\int_X f$ of continuous compactly-supported $V$-valued functions $f$ for $\mathrm{meas}\,(X) < +\infty$. Further, we prove that the desired integral lies in the (by hypothesis) *compact* set

$$\Big(\text{closure of convex hull of } f(X)\Big) \cdot \mathrm{meas}\,(X)$$

*Proof:* To simplify, divide by a constant to make $X$ have total measure 1. We may assume that $X$ is *compact* since the support of $f$ is compact. Let $H$ be the closure of the convex hull of $f(X)$ in $V$, compact by hypothesis. We will show that there is an integral of $f$ inside $H$.

For a *finite* subset $L$ of $V^*$, let

$$V_L \;=\; \{v \in V : \lambda v = \int_X \lambda \circ f,\; \forall \lambda \in L\}$$

And let

$$I_L = H \cap V_L$$

Since $H$ is compact and $V_L$ is closed, $I_L$ is *compact*. Certainly

$$I_L \cap I_{L'} \;=\; I_{L \cup L'}$$

for two finite subsets $L, L'$ of $V^*$. Thus, if we prove that all the $I_L$ are *non-empty*, then it will follow that the intersection of *all* these compact sets $I_L$ is non-empty. (This is the so-called *finite intersection property.*) That is, we will have *existence* of the integral.

To prove that each $I_L$ is non-empty for *finite* subsets $L$ of $V^*$, choose an ordering $\lambda_1, \ldots, \lambda_n$ of the elements of $L$. Make a continuous linear mapping $\Lambda = \Lambda_L$ from $V$ to $\mathbb{R}^n$ by

$$\Lambda(v) \;=\; (\lambda_1 v, \ldots, \lambda_n v)$$

Since this map is continuous, the image $\Lambda(f(X))$ is compact in $\mathbb{R}^n$.

For a finite set $L$ of functionals, the integral

$$y = y_L = \int_X \Lambda f(x)\; d\mu(x)$$

is readily defined by component-wise integration. Suppose that this point $y$ is in the convex hull of $\Lambda(f(X))$. Since $\Lambda_L$ is linear, $y = \Lambda_L v$ for some $v$ in the convex hull of $f(X)$. Then

$$\Lambda_L v \;=\; y \;=\; (\ldots, \int \lambda_i f(x)\, d\mu(X), \ldots)$$

Thus, the point $v$ lies in $I_L$ as desired. Granting that $y$ lies in the convex hull of $\Lambda_L(f(x))$, we are done.

To prove that $y = y_L$ as above lies in the convex hull of $\Lambda_L(f(X))$, suppose *not*. From the lemma below, in a *finite-dimensional* space the convex hull of a compact set is still compact, *without* having to take closure. Thus, invoking also the finite-dimensional case of the Hahn-Banach theorem, there would be a linear functional $\eta$ on $\mathbb{R}^n$ so that $\eta y > \eta z$ for all $z$ in this convex hull. That is, letting $y = (y_1, \ldots, y_n)$, there would be real $c_1, \ldots, c_n$ so that for all $(z_1, \ldots, z_n)$ in the convex hull

$$\sum_i c_i z_i \;<\; \sum c_i y_i$$

In particular, for all $x \in X$

$$\sum_i c_i \lambda_i(f(x)) \;<\; \sum_i c_i y_i$$

Integration of both sides of this over $X$ *preserves ordering*, giving the absurd

$$\sum_i c_i y_i < \sum_i c_i y_i$$

Thus, $y$ *does* lie in this convex hull. ///

**[6.0.1] Lemma:** The convex hull of a compact set $K$ in $\mathbb{R}^n$ is compact. In particular, we have compactness without taking closure.

*Proof:* We first claim that, for a set $E$ in $\mathbb{R}^n$ and for any $x$ a point in the convex hull of $E$, there are $n+1$ points $x_0, x_1, \ldots, x_n$ in $E$ of which $x$ is a convex combination.

By induction, to prove the claim it suffices to consider a convex combination $v = c_1 v_1 + \ldots + c_N v_N$ of vectors $v_i$ with $N > n+1$ and show that $v$ is actually a convex combination of $N-1$ of the $v_i$. Further, we can suppose without loss of generality that all the coefficients $c_i$ are non-zero.

Define a linear map

$$L \;:\; \mathbb{R}^N \longrightarrow \mathbb{R}^n \times \mathbb{R} \qquad \text{by} \qquad L(x_1, \ldots, x_N) \longrightarrow (\sum_i x_i v_i,\ \sum_i x_i)$$

By dimension-counting, since $N > n+1$ the kernel of $L$ must be non-trivial. Let $(x_1, \ldots, x_N)$ be a non-zero vector in the kernel.

Since $c_i > 0$ for every index, and since there are only finitely-many indices altogether, there is a constant $c$ so that $|cx_i| \le c_i$ for every index $i$, and so that $cx_{i_o} = c_{i_o}$ for at least one index $i_o$. Then

$$v \;=\; v - 0 \;=\; \sum_i c_i v_i - c \cdot \sum x_i v_i \;=\; \sum_i (c_i - cx_i) v_i$$

Since $\sum_i x_i = 0$ this is still a convex combination, and since $cx_{i_o} = c_{i_o}$ at least one coefficient has become zero. This is the induction, which proves the claim.

Using the previous claim, a point $v$ in the convex hull of $K$ is actually a convex combination $c_o v_o + \ldots + c_n v_n$ of $n+1$ points $v_o, \ldots, v_n$ of $K$. Let $\sigma$ be the compact set $(c_o, \ldots, c_n)$ with $0 \le c_i \le 1$ and $\sum_i c_i = 1$. The convex hull of $K$ is the image of the compact set

$$\sigma \times K^{n+1}$$

under the continuous map

$$L \;:\; (c_o, \ldots, c_n) \times (v_o, v_1, \ldots, v_n) \longrightarrow \sum_i c_i v_i$$

so is compact. This proves the lemma, finishing the proof of the theorem. ///

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# 7. *Historical notes and references*

Most investigation and use of integration of vector-valued functions is in the context of *Banach-space*-valued functions. Nevertheless, the idea of [Gelfand 1936] extended and developed by [Pettis 1938] immediately suggests a viewpoint not confined to the Banach-space case. A hint appears in [Rudin 1991].

This is in contrast to many of the more detailed studies and comparisons of varying notions of integral specific to the Banach-space case, such as [Bochner 1935]. A variety of developmental episodes and results in the Banach-space-valued case is surveyed in [Hildebrandt 1953]. Proofs and application of many of these results are given in [Hille-Phillips 1957]. (The first edition, authored by Hille alone, is sparser in this regard.) See also [Brooks 1969] to understand the viewpoint of those times.

One of the few exceptions to the apparent limitation to the Banach-space case is [Phillips 1940]. However, it seems that in the United States after the Second World War consideration of anything fancier than Banach spaces was not popular.

The present pursuit of the issue of quasi-completeness (and compactness of the closure of the convex hull of a compact set) was motivated originally by the discussion in [Rudin 1991], although the latter does not make clear that this condition is fulfilled in more than Fréchet spaces, and does not mention quasi-completeness. Imagining that these ideas must be applicable to distributions, one might cast about for means to prove the compactness condition, eventually hitting upon the hypothesis of quasi-completeness in conjunction with ideas from the proof of the Banach-Alaoglu theorem. Indeed, in [Bourbaki 1987] it is shown (by apparently different methods) that quasi-completeness implies this compactness condition, although there the application to vector-valued integrals is not mentioned. [Schaeffer-Wolff 1999] is a very readable account of further important ideas in topological vector spaces.

The fact that a bounded subset of a countable strict inductive limit of closed subspaces must actually be a bounded subset of one of the subspaces, easy to prove once conceived, is attributed to Dieudonne and Schwartz in [Horvath 1966]. See also [Bourbaki 1987], III.5 for this result. Pathological behavior of uncountable colimits was evidently first exposed in [Douady 1963].

Evidently *quotients* of quasi-complete spaces (by closed subspaces, of course) *may fail to be quasi-complete:* see [Bourbaki 1987], IV.63 exercise 10 for a construction.

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