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# *Banach-Alaoglu, boundedness, weak-to-strong principles*

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- Banach-Alaoglu: compactness of polars
- Variant Banach-Steinhaus (uniform boundedness)
- Bipolars
- Weak boundedness implies boundedness
- Weak-to-strong differentiability

The comparison of *weak* and *strong* differentiability is due to Grothendieck, although the original sources are not widely available.

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## 1. *Banach-Alaoglu theorem*

[1.0.1] **Definition:** The **polar** $U^o$ of an open neighborhood $U$ of 0 in a real or complex topological vector space $V$ is

$$U^o = \{\lambda \in V^* : |\lambda u| \le 1, \text{ for all } u \in U\}$$

[1.0.2] **Theorem:** *(Banach-Alaoglu)* In the weak star-topology on $V^*$ the polar $U^o$ of an open neighborhood $U$ of 0 in $V$ is *compact.*

*Proof:* For every $v$ in $V$ there is $t_v$ sufficiently large real such that $v \in t_v \cdot U$. Then $|\lambda v| \le t_v$ for $\lambda \in U^o$. Let

$$D_v = \{z \in \mathbb{C} : |z| \le t_v\}$$

The Tychonoff theorem yields the compactness of the product

$$P = \prod_{v \in V} D_v$$

Certainly

$$U^o \subset V^* \cup P$$

To prove the compactness of $U^o$ it suffices to show that the weak topology on $U^o$ is identical to the subspace topology inherited from $P$, and that $U^o$ is closed in $P$.

Regarding the topologies, the sub-basis sets

$$\{\lambda \in V^* : |\lambda v - \lambda_o v| < \delta\} \qquad \text{(for } v \in V \text{ and } \delta > 0)$$

for $V^*$ and

$$\{p \in P : |p_v - \lambda_o v| < \delta\} \qquad \text{(for } v \in V \text{ and } \delta > 0)$$

for $P$, respectively, have identical intersections with $U^o$. Thus, the weak star-topology on $U^o$ is the same as the product topology restricted to $U^o$.

To show that $U^o$ is closed, consider $L$ in the closure of $U^o$ in $P$. Given $x, y \in V$, $a, b \in \mathbb{C}$, the sets

$$\{p \in P : |(p - L)_x| < \delta\}$$

$$\{p \in P : |(p - L)_y| < \delta\}$$

$$\{p \in P : |(p-L)_{ax+by}| < \delta\}$$

are open in $P$ and contain $L$, so meet $U^o$. Let $\lambda \in U^o$ lie in the intersection of these three sets and $U^o$. Then

$$|aL_x + bL_y - L_{ax+by}| \le |a| \cdot |L_x - \lambda x| + |b| \cdot |L_y - \lambda y| + |L_{ax+by} - \lambda(ax+by)| + |a\lambda x + b\lambda y - \lambda(ax+by)|$$
$$\le |a| \cdot \delta + |b| \cdot \delta + \delta + 0 \qquad \text{(for every } \delta > 0)$$

so $L$ is linear.

Given $\varepsilon > 0$, let $N$ be a neighborhood of 0 in $V$ such that $x - y \in N$ implies

$$\lambda x - \lambda y \ \in \ N$$

Then

$$|L_x - L_y| \ = \ |L_x - \lambda x| + |L_y - \lambda y| + |\lambda x - \lambda y| \delta + \delta + \varepsilon$$

Thus, $L$ is continuous. Also,

$$|L_x - \lambda x| \ < \ \delta \qquad \text{(for all } x \in U \text{ and all } \delta > 0)$$

so $L \in U^o$. ///

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# 2. *Variant Banach-Steinhaus (uniform boundedness)*

This variant of the Banach-Steinhaus (uniform boundedness) theorem is used with Banach-Alaoglu to show that weak boundedness implies boundedness in a locally convex space, the starting point for *weak-to-strong principles.*

[2.0.1] **Theorem:** *(Variant Banach-Steinhaus)* Let $K$ be a compact convex set in a topological vectorspace $X$, and $\mathcal{T}$ a set of continuous linear maps $X \to Y$ from $X$ to another topological vectorspace $Y$. Suppose that for every *individual* $x \in K$ the collection of images

$$\mathcal{T}(x) = \{Tx : T \in \mathcal{T}\}$$

is a *bounded* subset of $Y$. Then there is a bounded set $B$ in $Y$ such that $\mathcal{T}(x) \subset B$ for *every* $x \in K$.

*Proof:* Let $B$ be the union

$$B \ = \ \bigcup_{x \in K} \mathcal{T}(x)$$

Let $U, V$ be balanced neighborhoods of 0 in $Y$ so that $\overline{U} + \overline{U} \subset V$, and let

$$E \ = \ \bigcap_{T \in \mathcal{T}} T^{-1}(\overline{U})$$

By the boundedness of $\mathcal{T}(x)$, there is a positive integer $n$ such that $\mathcal{T}(x) \subset nU$. Then $x \in nE$. For each $x \in K$ there is such $n$, so

$$K \ = \ \bigcup_n (K \cap nE)$$

Since $E$ is closed, the version of the Baire category theorem for locally compact Hausdorff spaces implies that at least one set $K \cap nE$ has non-empty interior in $K$.

For such $n$, let $x_o$ be an interior point of $K \cap nE$. Pick a balanced neighborhood $W$ of 0 in $X$ such that

$$K \cap (x_o + W) \subset nE$$

Since $K$ is compact, the set $K$ is bounded, so $K - x_o$ is bounded, so for large enough positive real $t$

$$K \subset x_o + tW$$

Since $K$ is convex, for any $x \in K$ and $t \geq 1$

$$(1 - t^{-1})x + t^{-1}x \in K$$

At the same time,

$$z - x_o = t^{-1}(x - x_o) \in W \qquad \text{(for large enough } t\text{)}$$

by the boundedness of $K$, so $z \in x_o + W$. Thus,

$$z \in K \cap (x_o + V) \subset nE$$

From the definition of $E$, $T(E) \subset \overline{U}$, so

$$T(nE) = nT(E) \subset n\overline{U}$$

And $x = tz - (t-1)x_o$ yields

$$Tx \in tn\overline{U} - (t-1)n\overline{U} \subset tn(\overline{U} + \overline{U})$$

by the balanced-ness of $U$. Since $\overline{U} + \overline{U} \subset V$, we have $B \subset tnV$. Since $V$ was arbitrary, this proves the boundedness of $B$. ///

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# 3. *Bipolars*

The **bipolar** $N^{oo}$ of an open neighborhood $N$ of 0 in a topological vector space $V$ is

$$N^{oo} = \{v \in V : |\lambda\, v| \leq 1 \text{ for all } \lambda \in N^o\}$$

where $N^o$ is the polar of $N$.

[3.0.1] **Proposition:** *(On bipolars)* Let $V$ be a locally convex topological vectorspace. Let $N$ be a convex and balanced neighborhood $N$ of 0. Then the bipolar $N^{oo}$ of $N$ is the closure $\overline{N}$ of $N$.

*Proof:* Certainly $N$ is contained in $N^{oo}$, and in fact $\overline{N}$ is contained in $N^{oo}$ since $N^{oo}$ is closed. By the local convexity of $V$, Hahn-Banach implies that for $v \in V$ but $v \notin \overline{N}$ there exists $\lambda \in V^*$ such that $\lambda v > 1$ and $|\lambda v'| \leq 1$ for all $v' \in \overline{N}$. Thus, $\lambda$ is in $N^o$, and every element $v \in N^{oo}$ is in $\overline{N}$, so $N^{oo} = \overline{N}$. ///

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# 4. *Weak boundedness implies strong boundedness*

[4.0.1] **Theorem:** Let $V$ be a locally convex topological vectorspace. A subset $E$ of $V$ is bounded if and only if it is weakly bounded.

*Proof:* That boundedness implies weak boundedness is trivial. On the other hand, suppose $E$ is weakly bounded, and let $U$ be a neighborhood of 0 in $V$ in the original topology. By local convexity, there is a convex (and balanced) neighborhood $N$ of 0 such that the closure $\overline{N}$ is contained in $U$.

By the weak boundedness of $E$, for each $\lambda \in V^*$ there is a bound $b_\lambda$ such that $|\lambda\, x| \leq b_\lambda$ for $x \in E$. By Banach-Alaoglu the polar $N^o$ of $N$ is compact in $V^*$. The functions $\lambda \to \lambda x$ are continuous, so by variant Banach-Steinhaus there is a uniform constant $b < \infty$ such that $|\lambda x| \leq b$ for $x \in E$ and $\lambda \in N^o$. Thus, $b^{-1}x$

is in the bipolar $N^{oo}$ of $N$, shown by the previous proposition to be the closure $\overline{N}$ of $N$. That is, $b^{-1}x \in \overline{N}$. By the balanced-ness of $N$,

$$E \subset t\overline{N} \subset tU \qquad \text{(for any } t > b)$$

This shows that $E$ is bounded. ///

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# 5. *Weak-to-strong differentiability*

Here is an application of the fact that weak boundedness implies boundedness. The result is well-known folklorically, for the case of Banach-space-valued functions, but the simple general case is generally treated as being apocryphal. In fact, weak-versus-strong differentiability and holomorphy were treated definitively in

[Grothendieck 1954] A. Grothendieck, *Espaces vectoriels topologiques*, mimeographed notes, Univ. Sao Paulo, Sao Paulo, 1954.

[Grothendieck 1953] A. Grothendieck, *Sur certains espaces de fonctions holomorphes, I, II*, J. Reine Angew. Math. **192** (1953), 35–64 and 77–95.

The first-mentioned of these is cited, for example, in

[Barros-Neto 1964] J. Barros-Neto, *Spaces of vector-valued real analytic functions*, Trans. AMS **112** (1964), 381–391.

[5.0.1] **Definition:** Let $f : [a, c] \to V$ be a $V$-valued function on an interval $[a, c] \subset \mathbb{R}$. The function $f$ is *differentiable* if for each $x_o \in [a, c]$

$$f'(x_o) = \lim_{x \to x_o} (x - x_o)^{-1} \Big( f(x) - f(x_o) \Big)$$

exists. The function $f$ is *continuously differentiable* if it is differentiable and if $f'$ is continuous. A $k$-times continuously differentiable function is said to be $C^k$, and a continuous function is said to be $C^o$.

[5.0.2] **Definition:** A $V$-valued function $f$ is **weakly** $C^k$ if for every $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ is $C^k$.

If there were any doubt, the present sense of *weak differentiability* of a function $f$ does not refer to distributional derivatives, but rather to differentiability of every scalar-valued function $\lambda \circ f$ where $f$ is vector-valued and $\lambda$ ranges over suitable continuous linear functionals.

[5.0.3] **Theorem:** Suppose that a function $f$ taking values in a quasi-complete locally convex topological vector space $V$, defined on an interval $[a, c]$, is *weakly* $C^k$. Then $f$ is *strongly* $C^{k-1}$.

First we need

[5.0.4] **Lemma:** Let $V$ be a quasi-complete locally convex topological vector space. Fix real numbers $a \le b \le c$. Let $f$ be a $V$-valued function defined on $[a, b) \cup (b, c]$. Suppose that for $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ extends to a $C^1$ function $F_\lambda$ on the whole interval $[a, c]$. Then $f(b)$ can be chosen such that the extended $f(x)$ is (strongly) continuous on $[a, c]$.

*Proof:* For each $\lambda \in V^*$, let $F_\lambda$ be the extension of $\lambda \circ f$ to a $C^1$ function on $[a, c]$. The differentiability of $F_\lambda$ implies that for each $\lambda$ the function

$$\Phi_\lambda(x, y) \;=\; \frac{F_\lambda(x) - F_\lambda(y)}{x - y}$$

has a continuous extension $\tilde{\Phi}_\lambda$ to the compact set $[a,c] \times [a,c]$. The image $C_\lambda$ of $[a,c] \times [a,c]$ under this continuous map is compact in $\mathbb{R}$, so bounded. Thus, the subset

$$\left\{ \frac{\lambda f(x) - \lambda f(y)}{x-y} : x \neq y \right\} \subset C_\lambda$$

is bounded in $\mathbb{R}$. Therefore, the set

$$E \;=\; \left\{ \frac{f(x) - f(y)}{x-y} : x \neq y \right\} \;\subset\; V$$

is *weakly* bounded, so $E$ is (strongly) bounded. Thus, for a balanced, convex neighborhood $N$ of 0 in $V$, there is $t_o$ such that $(f(x) - f(y))/(x-y) \in tN$ for $x \neq y$ in $[a,c]$ and $t \geq t_o$. That is,

$$f(x) - f(y) \;\in\; (x-y)tN$$

Thus, given $N$ and $t_o$ determined as above, for $|x-y| < \frac{1}{t_o}$

$$f(x) - f(y) \;\in\; N$$

That is, as $x \to 0$ the collection $f(x)$ is a bounded Cauchy net. By quasi-completeness, we can define $f(b) \in V$ as the limit of the values $f(x)$. For $x \to y$ the values $f(x)$ approach $f(y)$, so this extension of $f$ is continuous on $[a,c]$. ///

*Proof:* *(of theorem)* For $b \in (a,c)$ consider the function

$$g(x) = \frac{f(x) - f(b)}{x-b} \qquad\qquad \text{(for } x \neq b\text{)}$$

The assumed weak $C^2$-ness implies that every $\lambda \circ g$ extends to a $C^1$ function on $[a,c]$. By the lemma, $g$ itself has a continuous extension to $[a,c]$. In particular,

$$\lim_{x \to b} g(x)$$

exists, which exactly implies that $f$ is differentiable at $b$. Thus, $f$ is differentiable throughout $[a,c]$.

To prove the continuity of $f'$, consider again the function of two variables (initially for $x \neq y$)

$$g(x,y) = \frac{f(x) - f(y)}{x-y}$$

The weak $C^2$-ness of $f$ assures that $g$ extends to a weakly $C^1$ function on $[a,c] \times [a,c]$. In particular, the function $x \to g(x,x)$ of (the extended) $g$ is weakly $C^1$. This function is $f'(x)$. Thus, $f'$ is weakly $C^1$, so is (strongly) $C^o$.

Suppose that we already know that $f$ is $C^\ell$, for $\ell < k-1$. Consider the $\ell^{th}$ derivative $g = f^{(\ell)}$ of $f$. As $g$ is weakly $C^2$, it is (strongly) $C^1$ by the first part of the argument. That is, $f$ is $C^{\ell+1}$. ///

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