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Holomorphic vector-valued functions

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Abstract:

One of the first goals of a presentation of classical complex function theory is to prove Goursat's refinement of Cauchy's theorem: complex differentiability implies the conclusion of Cauchy's theorem, hence Cauchy's integral formula, hence complex analyticity (expandability in power series). Thereafter, one is typically very casual about terminology, using *complex differentiable* and *analytic* and *holomorphic* interchangeably. Indeed, use of the term *holomorphic* often signals the completion of this basic Cauchy theory.

Our goal here is to achieve the same effect for vector-valued functions. This does require a bit of rethinking power series with coefficients in topological vector spaces.

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1. Definition, examples

One of the first goals of a presentation of classical complex function theory is to prove Goursat's refinement of Cauchy's theorem: complex differentiability implies the conclusion of Cauchy's theorem, hence Cauchy's integral formula, hence complex analyticity (expandability in power series). Thereafter, one is typically very casual about terminology, using *complex differentiable* and *analytic* and *holomorphic* interchangeably. Indeed, use of the term *holomorphic* often signals the completion of this basic Cauchy theory.

Our goal here is to achieve the same effect for vector-valued functions. This does require a bit of rethinking power series with coefficients in topological vector spaces (see the Appendix).

Let V be a topological vector space and $f : D \to V$ be a V-valued function on an open set $D \subset \mathbf{C}$.

Definition: The function f is (strongly) **complex-differentiable** if, for all $z \in D$,

$$\lim_{w \to z} \frac{1}{w - z} \cdot (f(w) - f(z))$$

exists (in V).

Definition: The function f is (strongly) **analytic** if it is locally expressible as a convergent power series (with coefficients in V).

Definition: The function f is **weakly holomorphic** if, for all λ in the continuous dual V^*, the $\mathbf{C}$-valued function $\lambda \circ f$ is holomorphic in the classical sense.

Remark: It is a classical fact, Goursat's refinement of Cauchy's results, that complex-differentiable scalar-valued functions are, in fact, complex analytic (locally representable by convergent power series), and, thus, by Abel's theorem, certainly infinitely differentiable. Further, one has the Cauchy formulas, notions of *pole* versus *essential singularity*, Laurent expansions at poles, and so on. (Indeed, the analyticity is proven via the Cauchy theory.)

Theorem: For locally convex quasi-complete topological vector space V a weakly holomorphic V-valued function f is **strongly** holomorphic. And the usual Cauchy-theory integral formulas apply, for example

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

where γ is a closed path with z having winding number $+1$. And $f(z)$ is infinitely differentiable, in fact expressible as a convergent power series

$$f(z) = \sum_{n \geq 0} c_n \, (z - z_o)^n$$

with

$$c_n = n! \cdot f^n(z_o) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

Remark: The (strong) *continuity* follows without the quasi-completeness, but the formulation of Cauchy theory makes best sense if V is assumed quasi-complete (and locally convex).

Proof: First we show that weak holomorphy of f implies that $f : D \to V$ is (strongly) continuous (that is, in the original topology on V). Without loss of generality, we prove continuity at 0. We may also suppose that $f(0) = 0 \in V$. Let $\lambda \in V^*$. Since $\lambda \circ f$ is holomorphic and vanishes at 0, the function $(\lambda \circ f)(z)/z$ initially defined only on a punctured disk at 0 extends to a holomorphic function on a disk about 0. By Cauchy theory for scalar-valued holomorphic functions,

$$\frac{(\lambda \circ f)(z)}{z} = \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \cdot \frac{(\lambda \circ f)(\zeta)}{\zeta} \, d\zeta$$

where γ is a circle of radius $2r$ centered at 0, and $|z| < r$. Let M_λ be the sup of $|\lambda \circ f|$ on γ. Then the elementary estimate

$$\left| \frac{(\lambda \circ f)(z)}{z} \right| \leq \frac{1}{2\pi} \cdot (2\pi \cdot 2r) \cdot \frac{1}{r} \cdot \frac{M_\lambda}{2r} = \frac{M_\lambda}{r}$$

Thus, the set of values

$$\frac{f(z)}{z} : |z| \leq r$$

is weakly bounded (meaning that it is a bounded set when V is given the weak topology from V^*). But we know that weak boundedness implies (strong) boundedness, so this set is bounded. That is, given a balanced convex neighborhood N of 0 in V, there is $t_o > 0$ such that for complex w with $|w| \geq t_o$ that set of values lies inside wN. Then

$$f(z) \in zwN$$

and for $|z| < |w|$ we have $f(z) \in N$. We had taken $f(0) = 0$, so we have proven that, given N, for z sufficiently near 0

$$f(z) - f(0) \in N$$

This is (strong) continuity.

Now that we have the (strong) continuity, the rest of the argument is nearly obvious, keeping in mind properties of Gelfand-Pettis integrals. We certainly use the quasi-completeness for this. First, since $f(z)$ is now known to be (strongly) continuous, the integral

$$I(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

exists as a Gelfand-Pettis integral, and thus for any $\lambda \in V^*$

$$\lambda(I(z)) = \frac{1}{2\pi i} \int_\gamma \frac{(\lambda \circ f)(\zeta)}{\zeta - z} \, d\zeta = (\lambda \circ f)(z)$$

by the holomorphy of $\lambda \circ f$. Since linear functionals separate points, necessarily $I(z)$ is none other than $f(z)$, so we have the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

That is, the basic Cauchy formula is correct. However, notice that complex differentiability is not really immediate.

To prove complex differentiability of f at z_o, take $z_o = 0$ and use $f(0) = 0$, for convenience. Thus, there is a disk $|z| < 3r$ such that for every $\lambda \in V^*$

$$(\lambda \circ f)(z)/z$$

extends to a holomorphic function $F(z)$ on $|z| < r$. The continuity assures that the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta(\zeta - z)}\, d\zeta$$

exists, and by Cauchy theory for scalar-valued functions

$$\frac{(\lambda \circ f)(z)}{z} = (\lambda \circ F)(z) = \frac{1}{2\pi i} \int_\gamma \frac{(\lambda \circ f)(\zeta)}{\zeta(\zeta - z)}\, d\zeta$$

so since functionals separate points

$$\frac{f(z)}{z} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta(\zeta - z)}\, d\zeta$$

Now

$$\frac{1}{\zeta(\zeta - z)} = \frac{1}{\zeta^2} + \frac{z}{\zeta^2(\zeta - z)}$$

Thus,

$$\frac{f(z)}{z} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta^2}\, d\zeta + z \cdot \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta^2(\zeta - z)}\, d\zeta$$

Given a convex balanced neighborhood U of 0 in V, the set

$$K = \{f(\zeta) : |\zeta| = 2r\}$$

is compact, so contained in some multiple t_oU of U. Thus, for $|z| < r$,

$$\frac{f(z)}{z} - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta^2}\, d\zeta \in |z| \cdot \frac{1}{(2r)^2\, r} \cdot t_oU$$

Thus, as $z \to 0$ the limit of $f(z)/z$ exists. Since $f(0) = 0$, this proves the complex differentiability of f. We leave the derivation of the power series as an exercise in similar techniques. ///

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2. Appendix: Vector-valued power series, Abel's theorem

In this section V is a locally convex topological vector space, and we further assume that V is quasi-complete, so that (for example) Cauchy sequences in V converge.

Lemma: Let c_n be a *bounded* sequence of vectors in the locally convex quasi-complete topological vector space V. Let z_n be a sequence of complex numbers, let $0 \le r_n$ be real numbers such that $|z_n| \le r_n$, and suppose that $\sum_n r_n < +\infty$. Then the series

$$\sum_n c_n\, z_n$$

converges in V. Further, given a convex balanced neighborhood U of 0 in V let t be a positive real such that for all complex ζ with $|\zeta| \ge t$ we have $\{c_n\} \subset tU$. Then

$$\sum_n c_n\, z_n \;\in\; \left(\sum_n |z_n|\right) \cdot tU \subset \left(\sum_n r_n\right) \cdot tU$$

Proof: If N is a convex balanced neighborhood of 0 in the topological vector space and z and w are complex numbers with $|z| \le |w|$, then $zN \subset wN$, since $|z/w| \le 1$ implies $(z/w)N \subset N$, or $zN \subset wN$. Further, for an absolutely convergent series $\sum_n \alpha_n$ of complex numbers, for any n_o

$$\sum_{n \le n_o} (\alpha_n \cdot V) = \sum_{n \le n_o} (|\alpha_n| \cdot V) \subset \left(\sum_{n \le n_o} |\alpha_n| \right) \cdot N \subset \left(\sum_{n < \infty} |\alpha_n| \right) \cdot N$$

For a balanced open U containing 0, let t be large enough such that for any complex ζ with $|\zeta| \ge t$ the sequence c_n is contained in ζU. The previous discussion shows that

$$\sum_{m \le \ell \le n} c_\ell z_\ell \ \in \ (|z_m| + \ldots + |z_n|) \cdot tU$$

Given $\varepsilon > 0$, invoking absolute convergence, take m sufficiently large such that for all $n \ge m$

$$|z_m| + \ldots + |z_n| < t \cdot \varepsilon$$

Then

$$\sum_{m \le \ell \le n} c_\ell z_\ell \ \in \ t \cdot (\varepsilon / t) \cdot U = U$$

Thus, the original series is convergent. Since X is quasi-complete the limit exists in V. The last containment assertion follows from this discussion, as well. ///

Corollary: Let c_n be a bounded sequence of vectors in a locally convex quasi-complete topological vector space V. Then on $|z| < 1$ the series $f(z) = \sum_n c_n z^n$ converges and gives a holomorphic (infinitely-many times complex-differentiable) V-valued function.

Proof: The lemma shows that the series expressing $f(z)$ and its apparent k^{th} derivative $\sum_n c_n \binom{n}{k} z^{n-k}$ all converge for $|z| < 1$. The usual direct proof of Abel's theorem on the differentiability of (scalar-valued) power series can be adapted to prove the infinite differentiability of the X-valued function given by this power series, as follows. Let

$$g(z) = \sum_{n \ge 0} n c_n \, z^{n-1}$$

Then

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n \ge 1} c_n \left(\frac{z^n - w^n}{z - w} - n w^{n-1} \right)$$

For $n = 1$, the expression in the parentheses is 1. For $n > 1$, it is

$$\left(z^{n-1} + z^{n-2} w + \ldots + z w^{n-2} + w^{n-1} \right) - n w^{n-1}$$

$$= (z^{n-1} - w^{n-1}) + (z^{n-2} w - w^{n-1}) + \ldots + (z^2 w^{n-3} - w^{n-1}) + (z w^{n-2} - w^{n-1}) + (w^{n-1} - w^{n-1})$$

$$= (z - w) \left[(z^{n-2} + \ldots + w^{n-2}) + w(z^{n-3} + \ldots + w^{n-3}) + \ldots + w^{n-3}(z + w) + w^{n-2} + 0 \right]$$

$$= (z - w) \sum_{k=0}^{n-2} (k+1) \, z^{n-2-k} \, w^k$$

For $|z| \le r$ and $|w| \le r$ the latter expression is dominated by

$$|z - w| \cdot r^{n-2} \, \frac{n(n-1)}{2} < |z - w| \cdot n^2 \, r^{n-2}$$

Let U be a balanced neighborhood of 0 in X, and t a sufficiently large real number such that for all complex ζ with $|\zeta| \geq t$ all c_n lie in ζU. For $|z| \leq r < 1$ and $|w| \leq r < 1$, by the lemma,

$$\frac{f(z) - f(w)}{z - w} - g(w) = (z - w) \sum_{n \geq 2} c_n \cdot \left(\sum_{k=0}^{n-2} (k+1)\, z^{n-2-k}\, w^k \right) \in (z - w) \cdot \left(\sum_n n^2\, r^{n-2} \right) \cdot tU$$

Thus, for any given convex balanced neighborhood U of 0 in X, as $z \to w$

$$\frac{f(z) - f(w)}{z - w} - g(w)$$

eventually lies in U. ///

Corollary: Let c_n be a sequence of vectors in a Banach space X such that for some $r > 0$ the series $\sum |c_n| \cdot r^n$ converges in X. Then for $|z| < r$ the series $f(z) = \sum c_n z^n$ converges and gives a holomorphic (infinitely-many times complex-differentiable) X-valued function. ///
