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Hahn-Banach theorems

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- Continuous linear functionals
- Dominated Extension
- Separation Theorem
- Complex scalars
- Corollaries

The first point here is that convex sets can be *separated* by linear functionals. Second, continuous linear functionals on subspaces of a *locally convex* topological vectorspace have continuous extensions to the whole space.

Proofs are for *real* vectorspaces. The complex versions are corollaries.

A crucial corollary is that on locally convex topological vectorspaces continuous linear functionals *separate points*, meaning that for $x \neq y$ there is a continuous linear functional λ so that $\lambda(x) \neq \lambda(y)$. This separation property is essential in applications. Thus, the hypothesis of local convexity is likewise essential.

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1. *Continuous Linear Functionals*

Let k be $\mathbf{R}$ or $\mathbf{C}$ with usual absolute value, and V a k-vectorspace, without assumptions about topology on V for the moment. A k-linear k-valued function on V is a **linear functional**.

When V *has* a topology it makes sense to speak of **continuity** of functionals. The space of all continuous linear functionals on V is denoted V^*, suppressing reference to k.

A linear functional λ on V is *bounded* if there is a neighborhood U of 0 in V and constant c such that $|\lambda x| \leq c$ for $x \in U$. The following proposition is the general analogue of the assertion for Banach spaces, in which the *boundedness* has a different sense.

[1.0.1] **Proposition:** The following three conditions on a linear functional λ on a topological vectorspace V over k are equivalent:
- λ is continuous.
- λ is continuous at 0.
- λ is bounded.

Proof: The first implies the second. Assume the second. Given $\varepsilon > 0$, there is a neighborhood U of 0 such that $|\lambda|$ is bounded by ε on U. This proves boundedness in the topological vector space sense. Finally, suppose that $|\lambda x| \leq c$ on a neighborhood U of 0. Then, given $x \in V$ and given $\varepsilon > 0$, take

$$y \ \in \ x + \varepsilon \cdot U$$

With $x - y = \varepsilon \cdot u$ with $u \in U$,

$$|\lambda x - \lambda y| \ = \ \varepsilon |\lambda u| \leq \varepsilon \cdot c$$

Rewriting the argument replacing ε by a suitable multiple gives the desired result. ///

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2. *Dominated Extension*

In this section, vectorspaces are *real*.

The result involves elementary algebra and inequalities (apart from an invocation of transfinite induction) and is the heart of the matter. There is no direct discussion of topological vectorspaces. The goal is to *extend* a linear function while maintaining a comparison to another function (denoted p below). Thus, for this section we need *not* suppose the vectorspaces involved are *topological* vectorspaces.

Use of the term *extension* is standard, that a function $\widetilde{f}$ on a superset $\widetilde{X}$ of a set X is an *extension* of f if $\widetilde{f}$ restricted to the smaller set X is f.

Let V be a *real* vectorspace, without any assumption about topology. Let

$$p : V \to \mathbf{R}$$

be a *non-negative* real-valued function on V such that

$$\begin{cases} p(tv) = t \cdot p(v) & \text{(for } t \geq 0) \quad \textit{(positive-homogeneity)} \\ p(v+w) \leq p(v) + p(w) & \textit{(triangle inequality)} \end{cases}$$

Thus, p is not quite a *semi-norm*, lacking a description of $p(tv)$ for $t < 0$.

[2.0.1] **Theorem:** A real-linear function λ on a real vector subspace W of V such that

$$\lambda w \;\leq\; p(w) \qquad \text{(for all } w \in W)$$

has an extension to a real-linear function $\widetilde{\lambda}$ on all of V, such that

$$-p(-v) \;\leq\; \lambda v \;\leq\; p(v) \qquad \text{(for all } v \in V)$$

Proof: The crucial step is to extend the functional by a single step. That is, let $v \in V$. Attempt to define an extension $\widetilde{\lambda}$ of λ to $W + \mathbb{R}v$ by

$$\widetilde{\lambda}(w + tv) = \lambda w + ct$$

and see what conditions c must satisfy.

For all $w, w' \in W$

$$\lambda w - p(w-v) \;=\; \lambda(w+w') - \lambda w' - p(w-v)$$

$$\leq\; p(w+w') - \lambda(w') - p(w-v) = p(w - v + w' + v) - \lambda(w') - p(w-v)$$

$$\leq\; p(w-v) + p(w'+v) - \lambda w' - p(w-v) \;=\; p(w'+v) - \lambda w'$$

That is,

$$\lambda w - p(w-v) \;\leq\; p(w'+v) - \lambda w' \qquad \text{(for all } w, w' \in W)$$

Let σ be the sup of all the left-hand sides as w ranges over W. Since the right-hand side is finite, this sup is finite. Let μ be the inf of the right-hand side as w' ranges over W. We have

$$\lambda w - p(w-v) \;\leq\; \sigma \;\leq\; \mu \;\leq\; p(w'+v) - \lambda w'$$

Take any real number c such that

$$\sigma \;\leq\; c \;\leq\; \mu$$

and define

$$\widetilde{\lambda}(w+tv) \;=\; \lambda w + tc$$

To compare to p is easy: in the inequality

$$\lambda w - p(w-v) \;\leq\; \sigma$$

replace w by w/t with $t > 0$, multiply by t, and invoke positive-homogeneity to obtain

$$\lambda w - p(w - tv) \;\leq\; t\sigma$$

from which

$$\widetilde{\lambda}(w - tv) \;=\; \lambda w - tc \;\leq\; \lambda w - t\sigma \;\leq\; p(w - tv)$$

Likewise, from

$$\mu \;\leq\; p(w + v) - \lambda w$$

similarly

$$\widetilde{\lambda}(w + tv) \;=\; \lambda w + tc \;\leq\; \lambda w + t\mu \;\leq\; p(w + tv) \qquad (\text{for } t > 0)$$

which gives the other half of the desired inequality.

Thus,

$$\widetilde{\lambda}v \;\leq\; p(v) \qquad (\text{for all } v \in W + \mathbb{R}v)$$

Using the linearity of $\widetilde{\lambda}$,

$$\widetilde{\lambda}(v) \;=\; -\widetilde{\lambda}(-v) \;\geq\; -p(-v)$$

which gives the bottom half of the comparison of $\widetilde{\lambda}$ and p.

To extend to a functional on the *whole* space dominated by p is a typical exercise in transfinite induction, executed as follows. Let E be the collection of all extensions $\widetilde{\lambda}_X$ of λ to a subspace X of V, with $\widetilde{\lambda}_X$ dominated by p. Order these extensions by $(X_1, \widetilde{\lambda}_1) \leq (X_2, \widetilde{\lambda}_2)$ if $X_1 \subset X_2$ and $\widetilde{\lambda}_2|_X = \widetilde{\lambda}_2$. By Hausdorff Maximality, there is a *maximal* totally ordered subset E_o of E. Let

$$V' \;=\; \bigcup_{(X, \widetilde{\lambda}_X) \in E_o} X$$

be the ascending union of all the subspaces in E_o. Define a linear functional $\widetilde{\lambda}$ on this union: for $v \in V'$, take X such that $(X, \widetilde{\lambda}_X) \in E_o$ and $v \in X$ and define

$$\widetilde{\lambda}v \;=\; \backslash lamtil_X v$$

Varying the choice of $(X, \widetilde{\lambda}_X)$ does not affect the definition of $\widetilde{\lambda}$, because E_o is totally ordered.

It remains to check that V' is the whole space V. As usual, if not, then the first part of the proof would create an extension to a properly larger subspace, contradicting the maximality. ///

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3. *Separation*

All vectorspaces are *real.* Let V be a *locally convex* topological vectorspace, meaning that there is a local basis at $0 \in V$ of convex sets.

[3.0.1] **Theorem:** A non-empty convex open subset X of a locally convex topological vectorspace V can be *separated* from a non-empty convex set Y in V if $X \cap Y = \phi$, in the sense that there is a *continuous* real-linear real-valued functional λ on V and a constant c such that

$$\lambda x \;<\; c \;\leq\; \lambda y \qquad (\text{for all } x \in X \text{ and } y \in Y)$$

Proof: Fix $x_o \in X$ and $y_o \in Y$. Since X is open, $X - x_o$ is open, and so is

$$U = (X - x_o) - (Y - y_o) = \{(x - x_o) - (y - y_o) : x \in X, \;\; y \in Y\}$$

Since $x_o \in X$ and $y_o \in Y$, U contains 0. Since X, Y are convex, U is convex.

The **Minkowski functional** $p = p_U$ attached to U is defined to be

$$p(v) \;=\; \inf\{t > 0 : v \in tU\}$$

The convexity assures that p has the *positive-homogeneity* and *triangle-inequality* properties of the auxiliary functional p in the dominated extension theorem.

Let $z_o = -x_o + y_o$. Since $X \cap Y = \phi$, $z_o \notin U$, so $p(z_o) \geq 1$. Define a linear functional λ on $\mathbb{R}z_o$ by

$$\lambda(tz_o) \;=\; t$$

Check that λ is *dominated* by p in the sense of the dominated extension theorem:

$$\lambda(tz_o) \;=\; t \;\leq\; t \cdot p(z_o) \;=\; p(tz_o) \qquad\qquad (\text{for } t \geq 0)$$

while

$$\lambda(tz_o) \;=\; t < 0 \;\leq\; p(tz_o) \qquad\qquad (\text{for } t < 0)$$

Thus, indeed,

$$\lambda(tz_o) \;\leq\; p(tz_o) \qquad\qquad (\text{for all real } t)$$

Thus, λ extends to a real-linear real-valued functional λ on V, still such that

$$-p(-v) \;\leq\; \lambda v \;\leq\; p(v) \qquad\qquad (\text{for all } v \in V)$$

From the definition of p, $|\lambda| \leq 1$ on U. Thus, on $\frac{\varepsilon}{2}U$ we have $|\lambda| < \varepsilon$. That is, the linear functional λ is *bounded*, so is *continuous* at 0, so is *continuous* on V.

For arbitrary $x \in X$ and $y \in Y$,

$$\lambda x - \lambda y + 1 \;=\; \lambda(x - y + z_o) \;\leq\; p(x - y + z_o) \;<\; 1$$

since $x - y + z_o \in U$. Thus, for all such x, y,

$$\lambda x - \lambda y < 0$$

Therefore, $\lambda(X)$ and $\lambda(Y)$ are *disjoint* convex subsets of $\mathbb{R}$. Since λ is not the zero functional, it is *surjective*, and so is an *open* map. Thus, $\lambda(X)$ is open, and

$$\lambda(X) \;< \sup\; \lambda(X) \;\leq\; \lambda(Y)$$

as desired. ///

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4. *Complex scalars*

The *dominated extension* and *separation* theorems, stated and proven there for *real* vectorspaces, have analogues in the complex case, just corollaries of the the real-scalar results.

Let V be a complex vectorspace. Given a complex-linear complex-valued functional λ on V, let its real part be

$$\operatorname{Re}\lambda(v) \;=\; \frac{\lambda v + \overline{\lambda v}}{2}$$

where the overbar denotes complex conjugation. On the other hand, given a *real*-linear *real*-valued functional u on V, its *complexification* is

$$\mathrm{Cx}\, u(x) \;=\; u(x) - iu(ix) \qquad\qquad (\text{where } i = \sqrt{-1})$$

[4.0.1] Proposition: The complexification Cx u of a real-linear functional u on the complex vectorspace V is a complex-linear functional such that

$$\mathrm{Re}\,\mathrm{Cx}\, u \;=\; u$$

For a complex-linear functional λ

$$\mathrm{Cx}\,\mathrm{Re}\,\lambda \;=\; \lambda$$

Straightforward. ///

[4.0.2] Theorem: Let p be a *seminorm* on the complex vectorspace V and λ be a complex-linear function on a complex vector subspace W of V, such that

$$|\lambda w| \;\le\; p(w) \qquad\qquad (\text{for all } w \in W)$$

Then there is an extension of λ to a complex-linear function $\widetilde{\lambda}$ on V, such that

$$|\widetilde{\lambda} v| \;\le\; p(v) \qquad\qquad (\text{for all } v \in V)$$

Proof: Certainly if $|\lambda| \le p$ then $|\mathrm{Re}\,\lambda| \le p$. Then by the theorem for *real*-linear functionals, there is an extension u of $\mathrm{Re}\,\lambda$ to a *real*-linear functional u such that still $|u| \le p$. Let

$$\widetilde{\lambda} = \mathrm{Cx}\ u$$

All that remains to show, in light of the proposition above, is that $|\widetilde{\lambda}| \le p$.

To this end, for given $v \in V$, let μ be a complex number of absolute value 1 such that

$$|\widetilde{\lambda} v| \;=\; \mu \widetilde{\lambda} v$$

Then

$$|\widetilde{\lambda} v| \;=\; \mu\widetilde{\lambda} v \;=\; \widetilde{\lambda}(\mu v) \;=\; \mathrm{Re}\,\widetilde{\lambda}(\mu v) \;\le\; p(\mu v) \;=\; p(v)$$

using the seminorm property of p. Thus, the complex-linear functional $\widetilde{\lambda}$ made by complexifying the *real*-linear extension of the real part of λ satisfies the desired bound. ///

[4.0.3] Theorem: Let X be a non-empty convex open subset of a locally convex topological vectorspace V not meeting a non-empty convex set Y in V. Then there is a *continuous* complex-linear complex-valued functional λ on V and a constant c such that

$$\mathrm{Re}\,\lambda x \;<\; c \;\le\; \mathrm{Re}\,\lambda y \qquad\qquad (\text{for all } x \in X \text{ and } y \in Y)$$

Proof: Invoke the real-linear version of the theorem to make a *real*-linear functional u such that

$$ux \;<\; c \;\le\; uy \qquad\qquad (\text{for all } x \in X \text{ and } y \in Y)$$

By the proposition, u is the real part of its own complexification $\lambda = \mathrm{Cx}\, u$. ///

5. *Corollaries*

The corollaries hold for both real *or* complex scalars.

[5.0.1] **Corollary:** Let V be a locally convex topological vectorspace with K *compact* convex non-empty subset and C is a *closed* convex subset with $K \cap C = mtset$. Then there is a continuous linear functional λ on V and real constants $c_1 < c_2$ such that

$$\mathrm{Re}\,\lambda x \;\le\; c_1 \;<\; c_2 \;\le\; \mathrm{Re}\,\lambda y \qquad\qquad \text{(for all } x \in K \text{ and } y \in C\text{)}$$

Proof: Let U be a small-enough convex neighborhood of 0 in V such that

$$(K + U) \cap C \;=\; \phi$$

Apply the separation theorem to $X = K + U$ and $Y = C$. The constant c_2 can be taken to be $c_2 = \sup \mathrm{Re}\ \lambda(K + U)$. Since $\mathrm{Re}\,\lambda(K)$ is a compact subset of $\mathrm{Re}\,\lambda(K + U)$, its sup c_1 is strictly less than c_2. ///

[5.0.2] **Corollary:** Let V be a locally convex topological vectorspace, W a subspace, and $v \in V$. Let $\overline{W}$ denote the topological closure of W. Then $v \notin \overline{W}$ if and only if there is a *continuous* linear functional λ on V such that $\lambda(W) = 0$ while $\lambda(v) = 1$.

Proof: On one hand, if v lies in the closure of W, then any continuous function which is 0 on W must be 0 on v, as well.

On the other hand, suppose that v does *not* lie in the closure of W. Then apply the previous corollary with $K = \{v\}$ and $C = \overline{W}$. We find that

$$\mathrm{Re}\ \lambda(\{v\}) \cap \mathrm{Re}\ \lambda(\overline{W}) \;=\; \phi$$

Since $\mathrm{Re}\ \lambda(\overline{W})$ is a *proper* vector subspace of the real line, it must be $\{0\}$. Then $\mathrm{Re}\,\lambda v \neq 0$. Divide λ by the constant $\mathrm{Re}\,\lambda(v)$ to obtain a continuous linear functional zero on W but 1 on v. ///

[5.0.3] **Corollary:** Let V be a locally convex topological real vectorspace. Let λ be a continuous linear functional on a subspace W of V. Then there is a continuous linear functional λ on V extending λ.

Proof: Without loss of generality, take $\lambda \neq 0$. Let W_o be the kernel of λ (on W), and pick $w_1 \in W$ such that $\lambda w_1 = 1$. Evidently w_1 is not in the closure of W_o, so there is λ on the whole space V such that $\lambda|_{W_o} = 0$ and $\lambda w_1 = 1$. It is easy to check that this λ is an extension of λ. ///

[5.0.4] **Corollary:** Let V be a locally convex topological vectorspace. Given two distinct vectors $x \neq y$ in V, there is a continuous linear functional λ on V such that

$$\lambda x \;\neq\; \lambda y$$

Proof: The set $\{x\}$ is compact convex non-empty, and the set $\{y\}$ is closed convex non-empty. Apply a corollary above. ///