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Essential self-adjointness

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The following has been well understood for 70-120 years, or longer, naturally not in contemporary terminology.

The differential operator $T = \frac{d^2}{dx^2}$ on $L^2[a,b]$ or $L^2(\mathbb{R})$ is a prototypical natural *unbounded operator*. It is undeniably *not continuous* in the L^2 topology: on $L^2[0,1]$ the norm of $f(x) = x^n$ is $1/\sqrt{2n+1}$, and the second derivative of x^n is $n(n-1)x^{n-2}$, so

$$\text{operator norm } \frac{d^2}{dx^2} \text{ on } L^2[0,1] \;\geq\; \sup_{n\geq 1} \frac{n(n-1)\cdot \frac{1}{\sqrt{2n-3}}}{\frac{1}{\sqrt{2n+1}}} \;=\; +\infty$$

That is, $\frac{d^2}{dx^2}$ is not a L^2-bounded operator on *polynomials* on $[0,1]$, so has no bounded *extension*[1] to $L^2[0,1]$.

Nevertheless, the geometric structure of Hilbert spaces is extremely useful, especially the simple duality and adjoint phenomena. This motivates reconsideration of unbounded, not-everywhere-defined, but *densely* defined operators on Hilbert spaces. [2]

Let V be a Hilbert space, with hermitian inner product $\langle,\rangle$ and corresponding norm $|\cdot|$. Let T be an *unbounded* linear map $T : D_T \to V$ defined on a *dense* subspace D_T of V. We say that T is *on* V, even though its domain may be strictly smaller. We are interested in *symmetric* operators, meaning that

$$\langle Tv, w\rangle \;=\; \langle v, Tw\rangle \qquad \text{(for all } v, w \in D_T)$$

For unbounded operators, specification of the *domain* is critical. An operator $T : D_T \to V$ *extends* another operator $S : D_S \to V$ when $D_S \subset D_T$ and T agrees with S on D_S. This partial ordering on unbounded operators on V is written

$$S \;\subset\; T \qquad \text{(when } D_S \subset D_T \text{ and } T\big|_{D_S} = S)$$

In this notation, in terms of the *adjoint*[3] T^* of T,

$$\begin{cases} T \text{ symmetric} & \iff \quad T \subset T^* \\ T \text{ self-adjoint} & \iff \quad T = T^* \end{cases} \qquad \text{(for densely-defined } T)$$

[1] Whether or not the Axiom of Choice is used to artificially extend $\frac{d^2}{dx^2}$ to $L^2[0,1]$, that extension is not continuous, because the restriction to polynomials is already not continuous. The unboundedness/non-continuity is inescapable.

[2] Alternatively, one might allow more complicated topologies than that of a single Hilbert space, as did Friedrichs, Sobolev, Schwartz, and Grothendieck. In fact, a combination of approaches seems optimal.

[3] For an unbounded operator T, symmetric or not, to have a well-defined *adjoint* T^* requires the domain D_T be *dense*. As discussed in an appendix, the graph of the adjoint T^* is essentially the orthogonal complement of the graph of T. For T symmetric and densely-defined, the domain of T^* is dense.

We want *symmetric* extensions of symmetric operators.[4] The adjoint T^* of symmetric densely-defined T is an extension of T, but is *not* symmetric generally: symmetry of T^* would require $T^* = T^{**}$. We recall in an appendix that T^{**} is the *closure*[5] of T, and generally all that can be said is that

$$T \subset T^{**} \subset T^* \qquad \text{(for densely-defined, symmetric } T)$$

Since T^{**} is the closure of T, it *is* symmetric. But simple examples below demonstrate that T^{**} typically has strictly smaller domain than T^*, and then T^* is not symmetric, and T^{**} is not self-adjoint, since $(T^{**})^* = T^*$.

Thus, the issue of *self-adjoint extensions* of symmetric, densely-defined T on V cannot be settled trivially.

Further, we would prefer to be able to do *computations* with the self-adjoint extension, so an extension produced too non-constructively is of little values.

In practice, we are interested in *positive*, symmetric, densely-defined operators T, meaning that

$$\langle Tv, v\rangle \geq 0 \qquad \text{(for all } v \in D_T)$$

Friedrichs exhibited a *canonical* positive self-adjoint extension $T_{\text{Fr}} \supset T$ of a *positive*, symmetric, densely-defined operator, in a manner amenable to computation. [6]

However, the same simple examples that exhibit non-symmetric adjoints to symmetric operators show that there can be *many* self-adjoint extensions, *incomparable* in the partial ordering on operators.

Thus, we are interested clarifying conditions under which symmetric, densely-defined T has a *unique* self-adjoint extension. Such T is called **essentially self-adjoint**.

Again, as the examples below illustrate, it is unreasonable to expect even naturally-occurring positive, symmetric operators to have *unique* self-adjoint extensions. Nevertheless, situations in which the complication of non-uniqueness disappears are much simpler.

In brief, for unbounded operators arising from differential operators, imposition of different *boundary conditions* often gives rise to mutually incomparable self-adjoint extensions. Thus, *free-space* situations, lacking boundary conditions, are the best candidates for essential self-adjointness. This is illustrated in the following cautionary example.

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[4] It is easy to make non-symmetric extensions of not-everywhere-defined symmetric operators. For example, for S symmetric but not everywhere defined, take $w \in D_S$ with $|w| = 1$, and $0 \neq v \notin D_S$. Define an extension T of S by $Tv = c \cdot w$, for $c \in \mathbb{R}$. For T to be *symmetric* requires

$$c = c \cdot \langle w, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \langle v, Sw\rangle$$

[5] A possibly unbounded operator $T : D_T \to V$ is *closed* when its graph $\{x \oplus Tx : x \in D_T\}$ is closed in $V \oplus V$. The *closure* $\overline{T}$ is the operator *(if it exists)* whose graph is the closure of the graph of T. *Continuous* maps T have closed graphs. By the *Closed Graph Theorem*, an *everywhere-defined linear* map with closed graph is continuous. A not-everywhere-defined linear map can be closed without being continuous. As in the appendix, a *symmetric*, densely-defined, unbounded operator T has closure $\overline{T} = T^{**}$. Non-symmetric operators need not be *closable*.

[6] Friedrichs originally discussed *semi-bounded* symmetric operators T, meaning that there is a constant $C \in \mathbb{R}$ such that $\langle Tv, v\rangle \geq C \cdot \langle v, v\rangle$ for all $v \in D_T$. Everything reduces to the case $C = 0$, or, also, to $C = 1$.

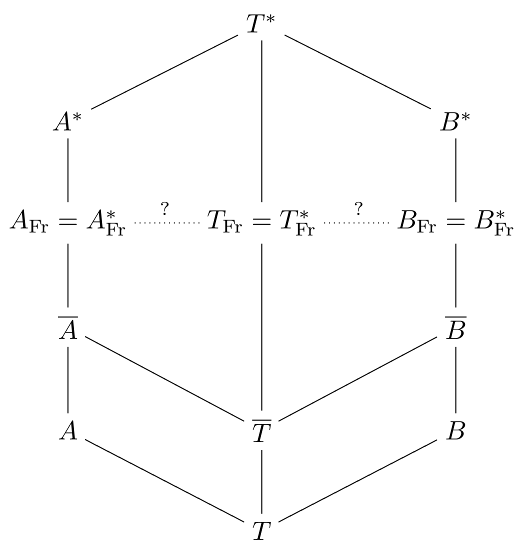
1. *Cautionary example*

[1.1] Non-symmetric adjoints of symmetric operators

Just below, *many different* positive, symmetric extensions of a natural, positive, symmetric, densely-defined operator T are exhibited, with *no two having a common symmetric extension.* This is not obviously possible.

In that situation, the closure $\overline{T} = T^{**}$ is *not* self-adjoint, equivalently, T^* is *not* symmetric, proven as follows.

Suppose positive, symmetric, densely-defined T has positive, symmetric extensions A, B *admitting no common symmetric extension.* Friedrichs' construction $T \to T_{\rm Fr}$ applies to T, A, B. The inclusion-reversing property [7] of $S \to S^*$ gives a diagram of extensions, where ascending lines indicate extensions:



Since T^* *is* a common extension of A, B, but A, B have no common *symmetric* extension, T^* cannot be symmetric. Thus, any such situation gives an example of *non-symmetric adjoints* of symmetric operators. Equivalently, $\overline{T}$ cannot be self-adjoint, because its adjoint is T^*, which cannot be *symmetric.*

Further, although $\overline{A}$ and $\overline{B}$ *are* (not necessarily *proper*) extensions of $\overline{T}$, neither of their Friedrichs extensions can be directly comparable to that of $\overline{T}$ without being *equal* to it, since comparable self-adjoint densely-defined operators are necessarily equal. [8] By hypothesis, A, B have no common symmetric extension, so it cannot be that *both* equalities hold.

[1.2] Example: symmetric extensions lacking a common symmetric extension

Let $V = L^2[a, b]$, $T = -d^2/dx^2$, with domain

$$D_T \;=\; \{f \in C^\infty[a, b] \;:\; f \text{ vanishes to infinite order at } a, b\}$$

The sign on the second derivative makes T *positive*: using the boundary conditions, integrating by parts,

$$\langle Tv, v\rangle \;=\; -\langle v'', v\rangle \;=\; -v'(b)\overline{v}(b) + v'(a)\overline{v}(a) + \langle v', v'\rangle \;=\; \langle v', v'\rangle \;\geq\; 0 \qquad \text{(for } v \in D_T)$$

[7] The domain-inclusion-reversing property of adjoint for *unbounded* but densely-defined operators is elementary, discussed in an appendix.

[8] A densely-defined self-adjoint operator cannot be a proper extensions of another such: for $S \subset T$ with $S = S^*$ and $T = T^*$, the inclusion-reversing property gives $T = T^* \subset S^* = S$.

Similarly, integration by parts *twice* proves *symmetry*:

$$\langle Tv, w\rangle \;=\; -\langle v'', w\rangle \;=\; -v'(b)\overline{w}(b) + v'(a)\overline{w}(a) + \langle v', w'\rangle \;=\; \langle v', w'\rangle$$

$$=\; v(b)\overline{w}'(b) - v(a)\overline{w}'(a) - \langle v, w''\rangle \;=\; \langle v, Tw\rangle \qquad \text{(for } v, w \in D_T\text{)}$$

For each pair α, β of complex numbers, we can define an extension $T_{\alpha,\beta} = -d^2/dx^2$ of T by taking a larger domain, by relaxing the boundary conditions in various ways:

$$D_{\alpha,\beta} \;=\; \{f \in C^\infty[a,b] \;:\; f(a) = \alpha \cdot f(b),\; f'(a) = \beta \cdot f'(b)\}$$

Integration by parts gives

$$\langle T_{\alpha,\beta} v, w\rangle \;=\; v'(b)\overline{w}(b) \cdot (1 - \beta\overline{\alpha}) + v(b)\overline{w}'(b) \cdot (1 - \alpha\overline{\beta}) \;+\; \langle v, T_{\alpha,\beta} w\rangle \qquad \text{(for } v, w \in D_{\alpha,\beta}\text{)}$$

The values $v'(b)$, $v(b)$, $w(b)$, and $w'(b)$ can be arbitrary, so the extension $T_{\alpha,\beta}$ is *symmetric* if and only if $\alpha\overline{\beta} = 1$, and in that case T is *positive*, since again

$$\langle T_{\alpha,\beta} v, v\rangle \;=\; -\langle v'', v\rangle \;=\; \langle v', v'\rangle \;\ge\; 0 \qquad \text{(for } \alpha\overline{\beta} = 1 \text{ and } v \in D_{\alpha,\beta}\text{)}$$

For two values α, α', taking $\beta = 1/\overline{\alpha}$ and $\beta' = 1/\overline{\alpha}$, for the symmetric extensions $T_{\alpha,\beta}$ and $T_{\alpha',\beta'}$ to have a *common symmetric extension* $\widetilde{T}$ requires that the domain of $\widetilde{T}$ include both $D_{\alpha,\beta} \cup D_{\alpha',\beta'}$. The integration by parts computation gives

$$\langle \widetilde{T} v, w\rangle \;=\; v'(b)\overline{w}(b) \cdot (1 - \beta\overline{\alpha}) + v(b)\overline{w}'(b) \cdot (1 - \alpha\overline{\beta}) \;+\; \langle v, T_{\alpha,\beta} w\rangle$$

$$=\; v'(b)\overline{w}(b)(1 - \beta\overline{\alpha}') + v(b)\overline{w}'(b) \cdot (1 - \alpha\overline{\beta}') \;+\; \langle v, \widetilde{T} w\rangle \qquad \text{(for } v \in D_{\alpha,\beta},\; w \in D_{\alpha',\beta'}\text{)}$$

Thus, the required symmetry $\langle \widetilde{T} v, w\rangle = \langle v, \widetilde{T} w\rangle$ holds only for $\alpha = \alpha'$ and $\beta = \beta'$. That is, the original operator T has a continuum of distinct symmetric extensions, no two of which admit a common symmetric extension.

In particular, no two of these symmetric extensions can have a common *self-adjoint* extension. Yet, each does have at least the *Friedrichs* positive, self-adjoint extension (see appendix). Thus, T has infinitely-many distinct positive, self-adjoint extensions.

Proliferation of mutually incompatible symmetric extensions of differential operators due to *boundary conditions* is typical. This correctly suggests that a symmetric differential operator more likely has a *unique* self-adjoint extension when there are no boundary conditions, that is, in a *free-space* problem.

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2. *Criterion for essential self-adjointness*

A symmetric, densely-defined operator is *essentially self-adjoint* when it has a *unique* self-adjoint extension.

Since a self-adjoint operator is *closed*, any self-adjoint extension of symmetric T must extend the closure $\overline{T}$.

The natural examples above exhibited positive, symmetric, densely-defined operators with *many* distinct positive self-adjoint extensions, so essential self-adjointness is not typical.

The usual argument (see appendix) shows that a symmetric T has no non-real complex eigenvalues λ, that is, that $T - \lambda$ is *injective* on D_T. This allows definition of an operator U on the image $(T - \lambda)D_T$ by

$$U \;=\; (T - \overline{\lambda})(T - \lambda)^{-1} \qquad \text{(for } \lambda \notin \mathbb{R}, \text{ on the image } (T - \lambda)D_T\text{)}$$

[2.0.1] Claim: The operator

$$U = (T - \overline{\lambda})(T - \lambda)^{-1} \qquad \text{(for } \lambda \notin \mathbb{R}\text{, on the image } (T-\lambda)D_T)$$

defined on the image $(T - \lambda)D_T$ is *unitary*, in the sense that $\langle Uv, Uw\rangle = \langle v, w\rangle$ for v, w in the domain of U.

Proof: For v, w in the image $(T - \lambda)D_T$, let $v' = (T - \lambda)^{-1}v$ and $w' = (T - \lambda)^{-1}w$. Then

$$\langle Uv, Uw\rangle = \langle (T - \overline{\lambda})v', (T - \overline{\lambda})w'\rangle$$

while

$$\langle v, w\rangle = \langle (T - \lambda)v', (T - \lambda)w'\rangle$$

Thus, we want to show that

$$\langle (T - \overline{\lambda})v', (T - \overline{\lambda})w'\rangle = \langle (T - \lambda)v', (T - \lambda)w'\rangle$$

This follows from the symmetry of T. ///

[2.0.2] Theorem: For *closed*, symmetric, densely-defined T, suppose that for some non-real λ both $(T - \lambda)D_T$ and $(T - \overline{\lambda})D_T$ are *dense*. Then T is *self-adjoint*.

Proof: First, claim that for *closed* and symmetric T, for non-real λ the image $(T - \lambda)D_T$ is *closed*. To see this, let $(T - \lambda)v_i$ be Cauchy, with v_i in the domain of U. By the unitariness of U, the sequence

$$U\big((T - \lambda)v_i\big) = (T - \overline{\lambda})v_i$$

is also Cauchy. Subtracting one sequence from the other, $(\lambda - \overline{\lambda})v_i$ is Cauchy. Since $\lambda \notin \mathbb{R}$, v_i is Cauchy. Similarly, adding the two sequences, $(2T + \lambda + \overline{\lambda})v_i$ is Cauchy. Because v_i is Cauchy, $(\lambda + \overline{\lambda})v_i$ is Cauchy, so $2Tv_i$ and Tv_i are Cauchy. Since the graph of T is closed, the sequence $v_i \oplus Tv_i$ converges to some $v \oplus Tv$ *in the graph of* T. Thus, $(T - \lambda)v_i$ certainly converges to $(T - \lambda)v$, and verifies the claim that $(T - \lambda)D_T$ is *closed*.

By hypothesis, the closed subspaces $(T - \lambda)D_T$ and $(T - \lambda)D_T$ are also *dense*, so each is the whole space V.

Given v in the domain D_{T^*} of the adjoint T^*, we show that $v \in D_T$. Since $(T - \lambda)D_T = V$, there is $v' \in D_T$ such that

$$(T - \lambda)v' = (T^* - \lambda)v$$

Thus,

$$\langle v', (T^* - \overline{\lambda})w\rangle = \langle (T - \lambda)v', w\rangle = \langle (T^* - \lambda)v, w\rangle = \langle v, (T - \overline{\lambda})w\rangle \qquad \text{(for all } w \in D_T)$$

Since $(T - \overline{\lambda})D_T$ is dense, $v' = v$. That is, $v \in D_T$. ///

[2.0.3] Corollary: For symmetric, densely-defined T, suppose that for some non-real λ both $(T - \lambda)D_T$ and $(T - \overline{\lambda})D_T$ are *dense*. Then the *closure* $\overline{T}$ of T is *self-adjoint*, and is the *unique* self-adjoint extension of T.

[2.0.4] Remark: As noted in an appendix, the closure $\overline{T}$ of T is the second adjoint T^{**}, and $\overline{T}^* = (T^{**})^* = T^*$.

Proof: The closure $\overline{T}$ extends T, and is *symmetric* for symmetric T. Certainly $(\overline{T} - \lambda)D_{\overline{T}}$ contains $(T - \lambda)D$, so when the latter is dense the former is dense. Thus, $\overline{T}$ meets the hypothesis of the theorem, and is self-adjoint.

Any self-adjoint extension $S = S^*$ of T is *closed*, since adjoints are closed. Thus, any self-adjoint extension S of T contains the closure $\overline{T} = T^{**}$, for topological reasons. Taking adjoints is inclusion-reversing, so:

$$S \;=\; S^* \;\subset\; (T^{**})^* \;=\; T^*$$

(The last equality is elementary, from characterization of adjoints in terms of graphs: see the appendix.) Therefore, $S = T^{**} = \overline{T}$. ///

[2.0.5] Remark: While many natural operators *are* essentially self-adjoint, the earlier example shows that many natural operators are *not*.

[2.0.6] Claim: For a symmetric, densely-defined operator T, the assertion that $(T - \lambda)D_T$ is *dense* is equivalent to the assertion that T^* does *not* have eigenvalue $\overline{\lambda}$.

[2.0.7] Remark: From the earlier examples, we know that T^* need not be *symmetric*, so the easy argument (as in the appendix) that eigenvalues of symmetric operators must be *real* does not apply.

Proof: The argument is the natural one. The density of $(T - \lambda)D_T$ implies that $\langle (T - \lambda)v, w\rangle = 0$ for all $v \in D_T$ if and only if $w = 0$. If $(T^* - \lambda)v = 0$, then

$$0 \;=\; \langle (T^* - \overline{\lambda})v, w\rangle \;=\; \langle v, (T - \lambda)w\rangle \qquad\qquad (\text{for all } w \in D_T)$$

Since $(T - \lambda)D_T$ is *dense*, this implies $w = 0$. Conversely, if $(T - \lambda)D_T$ were *not* dense, then its closure would not be the whole space, so would be orthogonal to some $v \neq 0$. Then

$$0 \;=\; \langle v, (T - \lambda)w\rangle \;=\; \langle (T^* - \overline{\lambda})v, w\rangle \qquad\qquad (\text{for every } w \in D_T, \text{ for } v \in D_{T^*})$$

Thus, since D_T is dense, we imagine that it would be consistent to *define* $T^*v = \overline{\lambda}v$. Indeed, since the graph of T^* is the orthogonal complement of the image of the graph of T under the isometry J (see appendix), there is no actual *choice*, and $T^*v = \overline{\lambda}v$. That is, if $(T-\lambda)D_T$ were not dense, then T^* would have eigenvalue $\overline{\lambda}$. ///

Thus, we have a variant form of the criterion for the closure of T being self-adjoint:

[2.0.8] Corollary: For symmetric, densely-defined T, suppose that for some non-real λ, neither λ nor $\overline{\lambda}$ is an *eigenvalue* for the adjoint T^*. Then the *closure* $\overline{T}$ of T is *self-adjoint*, and is the *unique* self-adjoint extension of T. ///

[2.0.9] Remark: In the situation of the corollary, since $\overline{T} = T^{**}$, and $T^{***} = T^*$, in fact $\overline{T}^* = T^*$.

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3. *Examples of essentially self-adjoint operators*

The cautionary examples above suggest that boundary conditions obstruct essential self-adjointness. Therefore, hoping for *unique* self-adjoint extensions suggests consideration of situations *without* boundary conditions. These are also called *free-space* problems.

These examples are in contrast to $L^2[a,b]$, treated above, where d^2/dx^2 on test functions has infinitely-many self-adjoint extensions with no common symmetric extension.

[3.1] Example: Laplacian on $V = L^2(\mathbb{R}/\mathbb{Z})$ Let $T = \frac{d^2}{dx^2}$ on the circle $G = \mathbb{R}/\mathbb{Z}$. Take as natural domain

$$D_T \;=\; V^\infty \;=\; C^\infty(G)$$

The fact that the smooth *vectors* in the representation are the smooth *functions* follows from Sobolev's imbedding. This also proves that V^∞ with its stronger topology is *complete.* [9]

The boundary terms in integration by parts vanish, and integration by parts twice proves the *symmetry* of T. A single integration by parts proves T is a negative (non-positive) operator, so the negative $-T$ of T is *positive*, and Friedrichs' construction applies. Thus, there is at least one *meaningful* self-adjoint extension. However, we want the *closure* $\overline{T}$ of T to be that self-adjoint extension, giving *uniqueness* in a strong, canonical fashion.

We do not directly characterize the domain D_{T^*} of T^*, apart from the fact that it contains the domain of T. It is convenient that T *stabilizes* D_T. The *translation action* of G on functions on G is

$$(R_x\, f)(y) \;=\; f(y+x)$$

This action is *unitary*, and gives a (jointly) continuous map

$$G \times V \;\longrightarrow\; V$$

An important feature of a *constant-coefficient* differential operator such as T is that it commutes with the translation action, at least on $D_T = V^\infty$: in symbols,

$$R_t \circ T \;=\; T \circ R_t \qquad\qquad (\text{for all } t \in G)$$

Indeed, this invariance allows such operators to descend from $\mathbb{R}$ to the quotient G. Certainly D_T is *stable* under translation.

[3.1.1] Claim: The domain D_{T^*} of T^* is *stable* under translation.

Proof: Let $J(x \oplus y) = -y \oplus x$ be the usual map on $V \oplus V$. The map J is an isometry with respect to the usual inner product

$$\langle x + x', y + y' \rangle \;=\; \langle x, y\rangle + \langle x', y'\rangle$$

on $V \oplus V$. The graph of the adjoint is characterized as the orthogonal complement of the image by J of the graph of T. Thus, for $y \oplus T^*y$ in the graph of T^*, for all $x \oplus Tx$ in the graph of T, because T commutes with R_t on D_T,

$$\langle R_t y \oplus R_t T^* y, J(x \oplus Tx)\rangle \;=\; \langle R_t y \oplus R_t T^* y, -Tx \oplus x\rangle$$

$$=\; \langle y \oplus T^* y, -R_t^{-1} Tx \oplus R_t^{-1} x\rangle \;=\; \langle y \oplus T^* y, -T R_t^{-1} x \oplus R_t^{-1} x\rangle \;=\; 0$$

because $R_t^{-1} x \in D_T$. Thus, $R_t y \in D_{T^*}$, as claimed. ///

For $\varphi \in C^\infty(G) = V^\infty$, there is an action of φ on V attached to the translation action, namely, the *integral operator*

$$R_\varphi v \;=\; \int_G \varphi(t) \cdot R_t\, v\; dt$$

Since $t \to \varphi(t) \cdot R_t v$ is a compactly-supported, continuous, V-valued function on G, it has a *Gelfand-Pettis* integral. Further, D_T is stable under this action. Indeed, the translation action

$$G \times V^\infty \;\longrightarrow\; V^\infty$$

[9] As usual, for non-negative integer ℓ, the ℓ^{th} L^2 Sobolev norm on $C_c^\infty(\mathbb{R})$ is given by a norm, $|f|_\ell^2 = |f|^2 + |f'|^2 + \ldots + |f^{(\ell)}|^2$, with L^2 norms-squared of f and its derivatives. The ℓ^{th} Sobolev space $\mathrm{Sob}(\ell)$ is the *completion* of $C_c^\infty(\mathbb{R})$ with respect to $|\cdot|_\ell$. Thus, the (filtered) limit $\mathrm{Sob}(+\infty) = \cap_\ell \mathrm{Sob}(\ell)$ is *complete*. It is immediate that smooth functions in $L^2(\mathbb{R})$ with all derivatives in $L^2(\mathbb{R})$ are in $\mathrm{Sob}(+\infty)$. That is, obviously $V^\infty \subset \mathrm{Sob}(+\infty)$. Sobolev's imbedding gives equality: $\mathrm{Sob}(+\infty) = V^\infty$.

is continuous with respect to the Fréchet-space topology on V^∞. For $\varphi \in C_c^\infty(G)$, the corresponding integral operator R_φ maps V to the Gårding subspace inside the smooth vectors V^∞. Again, the fact that the smooth *vectors* are smooth *functions* requires Sobolev's imbedding. [10]

[3.1.2] Claim: The operators R_φ for $\varphi \in C_c^\infty(G)$ commute with T^*.

Proof: Since the operators T on D_T and T^* on D_{T^*} are *not continuous* on V, the properties of Gelfand-Pettis integrals must be used scrupulously.

For $\varphi \in C_c^\infty(G)$, $v \in D_{T^*}$, and $w \in D_T$, using the commutativity of Gelfand-Pettis integrals with *continuous* maps, a sensible computation succeeds:

$$\langle R_\varphi T^* v, w\rangle \;=\; \langle \int_G \varphi(t) R_t T^* v\, dt,\; w\rangle \;=\; \int_G \langle \varphi(t) R_t T^* v, w\rangle\; dt \;=\; \int_G \varphi(t)\langle R_t T^* v, w\rangle\; dt$$

$$= \int_G \varphi(t)\langle T^* v, R_t^{-1} w\rangle\; dt \;=\; \int_G \varphi(t)\langle v, T\, R_t^{-1} w\rangle\; dt \;=\; \int_G \varphi(t)\langle v, R_t^{-1}\, Tw\rangle\; dt$$

$$= \int_G \varphi(t)\langle R_t v, Tw\rangle\; dt \;=\; \langle \int_G \varphi(t)\, R_t v\, dt,\; Tw\rangle \;=\; \langle R_\varphi v, Tw\rangle \;=\; \langle T^* R_\varphi v, w\rangle$$

This is the desired commutativity. ///

Now we can prove that T^* has *no* non-real eigenvalues, so T meets the hypotheses of the theorem of the previous section, and its closure $\overline{T}$ is the unique self-adjoint extension of T. Suppose $v \in D_{T^*}$ and $(T^* - \lambda)v = 0$. Then, for any $\varphi \in C_c^\infty(G)$,

$$0 \;=\; R_\varphi \cdot 0 \;=\; R_\varphi(T^* - \lambda)v \;=\; (T^* - \lambda)R_\varphi v \;=\; (T - \lambda)R_\varphi v$$

the last equality because R_φ maps everything to V^∞, on which T^* acts by T. Although T^* is *not* assured to be symmetric (unless $T^* = \overline{T} = T^{**}$, which is the sought-after essential self-adjointness of T itself!), the operator T *is* symmetric, so has no non-real eigenvalues, giving $R_\varphi v = 0$. For given φ, taking φ sufficiently far along in an *approximate identity* gives $R_\varphi v \neq 0$ for $v \neq 0$. Thus, we conclude that $v = 0$, and T^* has no non-real eigenvalues. ///

Thus, the closure $\overline{T}$ of T is the unique self-adjoint extension of T. Restricted to the graph of T, the metric on $V \oplus V$ gives norm-squared

$$|x \oplus Tx|^2 \;=\; |x|^2 + |Tx|^2 \;\geq\; |x|^2 + |\langle Tx, x\rangle| + |Tx|^2 - \tfrac{1}{2}\cdot(|x| + |Tx|)^2 \;\geq\; \tfrac{1}{2}\cdot\big(|x|^2 + \langle -Tx, x\rangle + |Tx|^2\big)$$

since Cauchy-Schwarz-Bunyakowsky and $2ab \leq (a+b)^2$ give

$$|\langle Tx, x\rangle| \;\leq\; |Tx|\cdot|x| \;\leq\; \tfrac{1}{2}\cdot(|x| + |Tx|)^2$$

The completion V_2 of D_T with respect to the norm attached to the hermitian inner product

$$\langle x, y\rangle_2 \;=\; \langle x, y\rangle + \langle -Tx, y\rangle + \langle T^2 x, y\rangle \qquad\qquad (\text{for } x, y \in D_T)$$

is exactly the domain of $\overline{T}$. In the $\langle,\rangle_2$ topology, T is *continuous* on D_T, and $\overline{T}$ is the *extension by continuity* to V_2.

[3.2] Example: Laplacian on $V = L^2(\mathbb{R})$ Let $T = \frac{d^2}{dx^2}$ on $G = \mathbb{R}$. Most of the discussion for $L^2[\mathbb{R}/\mathbb{Z}]$ applies here, as will. However, unlike the previous example, there are several plausible choices of domain

[10] As an example where smooth vectors are *not* smooth functions: in the translation representation of G on *distributions* on the circle, *all* distributions are smooth *vectors*, but most are not smooth *functions*.

D_T, many of them problematical. The crucial point in the argument for essential self-adjointness is that integral operators attached to test functions on the group map D_{T^*} to D_T. This dictates taking D_T to be the *smooth vectors* V^∞ in the representation:

$$D_T \;=\; V^\infty \;=\; \{f \in V : g \to R_g f \text{ is a smooth } V\text{-valued function}\}$$

By Sobolev, the space of smooth vectors consists of smooth *functions*:

$$D_T \;=\; \{f \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \;:\; f^{(i)} \in L^2(\mathbb{R}) \text{ for all } i\}$$

This also proves *completeness* of V^∞ in the Sob$(+\infty)$-topology.

[3.2.1] Claim: For f, g in V^∞, the boundary terms in integration by parts *vanish*, so *one* integration by parts gives non-positivity

$$\langle Tf, f\rangle \;=\; -\langle f', f'\rangle \;\le\; 0$$

and *two* integrations by parts gives symmetry

$$\langle Tf, g\rangle \;=\; \langle f, Tg\rangle \qquad\qquad (\text{for } f, g \in V^\infty)$$

Proof: The claim would follow from vanishing of the boundary terms in integration by parts, which would follow from vanishing at infinity of functions f in V^∞ and their derivatives f'. However, since square-integrable smooth functions need *not* go to 0 at infinity, some work is required.

On $\mathbb{R}$, the standard results about Sobolev spaces include the density of $C_c^\infty(\mathbb{R})$ in Sob$(+\infty) = \bigcap_{t\ge 0}$ Sob(t). Thus, extension by continuity gives the integration by parts identity on V^∞, giving symmetry and non-positivity. ///

[3.2.2] Remark: By the non-positiveness, the Friedrichs construction applies, so there is *at least one* meaningful self-adjoint extension. However, we want the *closure* $\overline{T}$ of T to be that self-adjoint extension, incidentally giving *uniqueness*.

Suppose $v \ne 0$ is a λ-eigenvector for T^* with non-real λ. We need no more information about the domain D_{T^*} of T^* than that it contains the domain of T, $D_T = V^\infty$. It is also convenient that T *stabilizes* D_T. The *translation action* of $\mathbb{R}$ on functions on $\mathbb{R}$ is

$$(R_x f)(y) \;=\; f(y + x)$$

This action is *unitary*, and gives a (jointly) continuous map

$$\mathbb{R} \times L^2(\mathbb{R}) \;\longrightarrow\; L^2(\mathbb{R})$$

An important feature of a *constant-coefficient* differential operator such as T is that it commutes with the translation action, at least on $D_T = C^\infty(\mathbb{R})$: in symbols,

$$R_t \circ T \;=\; T \circ R_t \qquad\qquad (\text{for all } t \in \mathbb{R})$$

Certainly D_T is *stable* under the translation action.

[3.2.3] Claim: The domain D_{T^*} of T^* is stable under translation action.

Proof: Let $J(x \oplus y) = -y \oplus x$ be the usual map on $V \oplus V$. The map J is an isometry with respect to the usual inner product

$$\langle x + x', y + y'\rangle \;=\; \langle x, y\rangle + \langle x', y'\rangle$$

on $V \oplus V$. The graph of the adjoint is characterized as the orthogonal complement of the image by J of the graph of T. Thus, for $y \oplus T^*y$ in the graph of T^*, for all $x \oplus Tx$ in the graph of T, because T commutes with R_t on D_T,

$$\langle R_t y \oplus R_t T^* y, J(x \oplus Tx)\rangle \;=\; \langle R_t y \oplus R_t T^* y, -Tx \oplus x\rangle$$

$$=\; \langle y \oplus T^* y, -R_t^{-1} Tx \oplus R_t^{-1} x\rangle \;=\; \langle y \oplus T^* y, -T R_t^{-1} x \oplus R_t^{-1} x\rangle$$

Thus, $R_t y \in D_{T^*}$, as claimed. ///

Test functions $\varphi \in C_c^\infty(\mathbb{R})$ act on the Hilbert space V and on the Fréchet space V^∞ by the *integral operator*

$$R_\varphi v \;=\; \int_{\mathbb{R}} \varphi(t) \cdot R_t\, v\, dt$$

Since $t \to \varphi(t) \cdot R_t v$ is a compactly-supported, continuous, V-valued or V^∞-valued function on $\mathbb{R}$, it has a *Gelfand-Pettis* integral. In particular, the domain $D_T = V^\infty$ is stable under this action.

[3.2.4] Claim: The operators R_φ for $\varphi \in C_c^\infty(G)$ commute with T^*.

Proof: Since the operators T on D_T and T^* on D_{T^*} are *not continuous* on V, the properties of Gelfand-Pettis integrals must be used carefully.

For $\varphi \in C_c^\infty(G)$, $v \in D_{T^*}$, and $w \in D_T$, using the commutativity of Gelfand-Pettis integrals with *continuous* maps,

$$\langle R_\varphi T^* v, w\rangle \;=\; \langle \int_G \varphi(t) R_t T^* v\, dt,\; w\rangle \;=\; \int_G \langle \varphi(t) R_t T^* v, w\rangle\; dt \;=\; \int_G \varphi(t) \langle R_t T^* v, w\rangle\; dt$$

$$=\; \int_G \varphi(t) \langle T^* v, R_t^{-1} w\rangle\; dt \;=\; \int_G \varphi(t) \langle v, T\, R_t^{-1} w\rangle\; dt \;=\; \int_G \varphi(t) \langle v, R_t^{-1}\, Tw\rangle\; dt$$

$$=\; \int_G \varphi(t) \langle R_t v, Tw\rangle\; dt \;=\; \langle \int_G \varphi(t)\, R_t v\, dt,\; Tw\rangle \;=\; \langle R_\varphi v, Tw\rangle \;=\; \langle T^* R_\varphi v, w\rangle$$

This is the desired commutativity. ///

Now we can prove that T^* has *no* non-real eigenvalues, so T meets the hypotheses of the theorem of the previous section, and its closure $\overline{T}$ is the unique self-adjoint extension of T. The pattern of symbols is the same as for $L^2(\mathbb{R}/\mathbb{Z})$. Suppose $v \in D_{T^*}$ and $(T^* - \lambda)v = 0$. Then, for any $\varphi \in C_c^\infty(G)$,

$$0 \;=\; R_\varphi \cdot 0 \;=\; R_\varphi (T^* - \lambda) v \;=\; (T^* - \lambda) R_\varphi v \;=\; (T - \lambda) R_\varphi v$$

the last equality because R_φ maps the whole space V to V^∞, on which T^* acts by T. Although T^* is *not* assured to be symmetric (unless $T^* = \overline{T} = T^{**}$, which is the sought-after essential self-adjointness of T itself!), the operator T *is* symmetric, so has no non-real eigenvalues, giving $R_\varphi v = 0$. For given φ, taking φ sufficiently far along in an *approximate identity* gives $R_\varphi v \neq 0$ for $v \neq 0$. Thus, we conclude that $v = 0$, and T^* has no non-real eigenvalues. ///

Thus, the closure $\overline{T}$ of T is the unique self-adjoint extension of T.

[3.2.5] Remark: For G a semi-simple real Lie group, Γ a reasonable discrete subgroup, K a maximal compact subgroup, Ω the Casimir element in the universal enveloping algebra $U\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of G, essentially the same argument applies to the operator Ω on $L^2(\Gamma\backslash G/K)$, as soon as we have a suitable *global* Sobolev theory.

That is, the action of Ω on $L^2(\Gamma\backslash G/K)$ is essentially self-adjoint.

4. *Appendix: Friedrichs' canonical self-adjoint extensions*

We recall Friedrichs' construction of canonical *self-adjoint* positive extensions of positive, symmetric, densely-defined operators.

[4.0.1] Theorem: (Friedrichs) A *positive*, densely-defined, symmetric operator T, D has a positive *self-adjoint* extension.

Proof: [11] Define a new hermitian form $\langle , \rangle_1$ and corresponding norm $\| \cdot \|_1$ by

$$\langle v, w \rangle_1 \;=\; \langle v, w \rangle + \langle Tv, w \rangle \qquad \text{(for } v, w \in D\text{)}$$

The symmetry and non-negativity of T make this positive-definite hermitian on D. Note that $\langle v, w \rangle_1$ makes sense whenever at least one of v, w is in D.

Let V_1 be the closure in V of D with respect to the metric d induced by $\| \cdot \|$. We claim that V_1 is also the d-completion of D. Indeed, for v_i a d-Cauchy sequence in D, v_i is Cauchy in V in the original topology, since

$$|v_i - v_j| \;\le\; |v_i - v_j|_1$$

For two sequences v_i, w_j with the same d-limit v, the d-limit of $v_i - w_i$ is 0. Thus,

$$|v_i - w_i| \;\le\; |v_i - w_i|_1 \;\longrightarrow\; 0$$

For $h \in V$ and $v \in V_1$, the functional $\lambda_h : v \to \langle v, h \rangle$ has a bound

$$|\lambda_h v| \;\le\; |v| \cdot |h| \;\le\; |v|_1 \cdot |h|$$

Thus, the norm of the functional λ_h on V_1 is at most $|h|$. By Riesz-Fischer, there is unique Bh in the Hilbert space V_1 with $|Bh|_1 \le |h|$, such that

$$\lambda_h v \;=\; \langle Bh, v \rangle_1 \qquad \text{(for } v \in V_1\text{)}$$

Thus,

$$|Bh| \;\le\; |Bh|_1 \;\le\; |h|$$

The map $B : V \to V_1$ is verifiably linear. There is an obvious *symmetry* of B:

$$\langle Bv, w \rangle \;=\; \lambda_w Bv \;=\; \langle Bv, Bw \rangle_1 \;=\; \overline{\langle Bw, Bv \rangle_1} \;=\; \overline{\lambda_v Bw} \;=\; \overline{\langle Bw, v \rangle} \;=\; \langle v, Bw \rangle \qquad \text{(for } v, w \in V\text{)}$$

Positivity of B is similar:

$$\langle Bv, v \rangle \;=\; \lambda_v Bv \;=\; \langle Bv, Bv \rangle_1 \;\ge\; \langle Bv, Bv \rangle \ge 0$$

Finally B is *injective*: if $Bw = 0$, then for all $v \in V_1$

$$0 \;=\; \langle v, 0 \rangle_1 \;=\; \langle v, Bw \rangle_1 \;=\; \lambda_w v \;=\; \langle v, w \rangle$$

Since V_1 is dense in V, $w = 0$. Similarly, if $w \in V_1$ is such that $\lambda_v w = 0$ for all $v \in V$, then $0 = \lambda_w w = \langle w, w \rangle$ gives $w = 0$. Thus, $B : V \to V_1$ is bounded, symmetric, positive, injective, with dense image. In particular, B is self-adjoint.

[11] We essentially follow [Riesz-Nagy 1955], pages 329-334.

Thus, B has a possibly *unbounded* positive, symmetric inverse A. Since B injects V to a dense subset V_1, necessarily A *surjects* from its domain (inside V_1) to V. We claim that A is *self-adjoint*. Let $S : V\oplus V \to V\oplus V$ by $S(v \oplus w) = w \oplus v$. Then

$$\text{graph } A \;=\; S(\text{graph } B)$$

Also, in computing orthogonal complements $X^\perp$, clearly

$$(S\,X)^\perp \;=\; S\big(X^\perp\big)$$

From the obvious $J \circ S = -S \circ J$, compute

$$\text{graph } A^* \;=\; (J \text{ graph } A)^\perp \;=\; (J \circ S \text{ graph } B)^\perp \;=\; (-S \circ J \text{ graph } B)^\perp$$

$$=\; -S\big((J \text{ graph } B)^\perp\big) \;=\; -\,\text{graph}\, A \;=\; \text{graph } A$$

since the domain of B^* is the domain of B. Thus, A is self-adjoint.

We claim that for v in the domain of A, $\langle Av, v\rangle \geq \langle v, v\rangle$. Indeed, letting $v = Bw$,

$$\langle v, Av\rangle \;=\; \langle Bw, w\rangle \;=\; \lambda_w Bw \;=\; \langle Bw, Bw\rangle_1 \;\geq\; \langle Bw, Bw\rangle \;=\; \langle v, v\rangle$$

Similarly, with $v' = Bw'$, and $v \in V_1$,

$$\langle v, Av'\rangle \;=\; \langle v, w'\rangle \;=\; \lambda_{w'} v \;=\; \langle v, Bw'\rangle_1 \;=\; \langle v, v'\rangle_1 \qquad (v \in V_1,\ v' \text{ in the domain of } A)$$

Since B maps V to V_1, the domain of A is contained in V_1. We claim that the domain of A is dense in V_1 in the d-topology, not merely in the subspace topology from V. Indeed, for $v \in V_1$ $\langle,\rangle_1$-orthogonal to the domain of A, for v' in the domain of A, using the previous identity,

$$0 \;=\; \langle v, v'\rangle_1 \;=\; \langle v, Av'\rangle$$

Since B injects V to V_1, A surjects from its domain to V. Thus, $v = 0$.

Last, prove that A is an extension of $S = 1_V + T$. On one hand, as above,

$$\langle v, Sw\rangle \;=\; \lambda_{Sw} v \;=\; \langle v, BSw\rangle_1 \qquad (\text{for } v, w \in D)$$

On the other hand, by definition of $\langle,\rangle_1$,

$$\langle v, Sw\rangle \;=\; \langle v, w\rangle_1 \qquad (\text{for } v, w \in D)$$

Thus,

$$\langle v, w - BSw\rangle_1 \;=\; 0 \qquad (\text{for all } v, w \in D)$$

Since D is d-dense in V_1, $BSw = w$ for $w \in D$. Thus, $w \in D$ is in the range of B, so is in the domain of A, and

$$Aw \;=\; A(BSw) \;=\; Sw$$

Thus, the domain of A contains that of S and extends S. ///

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5. *Appendix: graphs, closures, adjoints*

We recall some simple background material.

For a Hilbert space V, the direct sum $V \oplus V$ is a Hilbert space, with natural inner product

$$\langle v \oplus v', w \oplus w'\rangle \;=\; \langle v, v'\rangle + \langle w, w'\rangle$$

Define an isometry J of $V \oplus V$ by

$$J : V \oplus V \longrightarrow V \oplus V \qquad \text{by} \qquad v \oplus w \longrightarrow -w \oplus v$$

The adjoint T^* of T is characterized by its graph, which is the orthogonal complement in $V \oplus V$ to an image of the graph of T under J:

$$\text{graph } T^* = \text{orthogonal complement of } J(\text{graph } T)$$

Since T is densely-defined, for given $w \in V$ there is *at most* one possible value w' such that $w \oplus w' \in X$, so this orthogonality condition determines a well-defined function T^* on a subset of V, by

$$T^* w = w' \qquad \text{(if there exists } w' \in V \text{ such that } w \oplus w' \in X)$$

The linearity of T^* is immediate. Orthogonal complements are closed, so T^* has a closed graph.

This characterization of adjoint shows that, for $T_1 \subset T_1$ with dense domains, $T_1^* \subset T_1^*$, and $T_1 \subset T_1^{**}$.

In particular, for self-adjoint $S \subset T$,

$$T = T^* \subset S^* = S \qquad \text{(when } S \subset T \text{ are self-adjoint)}$$

so $S = T$. That is, distinct self-adjoint operators are *mutually incomparable.*

[5.0.1] Proposition: Eigenvalues for symmetric operators T, D are *real.*

Proof: This is the expected computation. Suppose $0 \neq v \in D$ and $Tv = \lambda v$. Then

$$\lambda\langle v,v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, T^*v\rangle \qquad \text{(because } v \in D \subset D^*)$$

Because T^* agrees with T on D,

$$\langle v, T^*v\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\bar{v}, v\rangle$$

Thus, λ is real. ///

[5.0.2] Definition: A densely-defined symmetric operator T, D is *positive* (or *non-negative*) when

$$\langle Tv, v\rangle \geq 0 \qquad \text{(for all } v \in D)$$

Certainly all the eigenvalues of a positive operator are non-negative real.

[5.1] Closures of symmetric operators

To make a closed operator extending a *symmetric*[12] operator T, the *closure* of (the graph of) T is the graph of T^{**}, seen as follows.

The graph of the adjoint L^* of a densely-defined linear map L is

$$\text{graph}\, L^* = \big(J(\text{graph}\, L)\big)^\perp$$

Thus, the *adjoint* of a densely-defined operator is always *closed.* Similarly, the closure of the graph of T is the graph of T^{**}:

$$\text{graph}\, T^{**} = \big(J(\text{graph}\, T^*)\big)^\perp = \big(J(J(\text{graph}\, T)^\perp)\big)^\perp = (\text{graph}\, T)^{\perp\perp} = \text{closure}(\text{graph}\, T)$$

Thus, generally, for symmetric T,

$$T \subset T^{**} \subset T^*$$

and T^{**} is closed. Taking adjoints again,

$$\text{graph}\, T^{***} = \big(J(\text{graph}\, T^{**})\big)^\perp = \big(J(J(\text{graph}\, T^*)^\perp)\big)^\perp = (\text{graph}\, T^*)^{\perp\perp}$$
$$= \text{closure}(\text{graph}\, T^*) = \text{graph}\, T^*$$

Thus,

$$T^{***} = T^*$$

[12] For not-necessarily symmetric operators T, the closure of the graph of T need not be the graph of an operator.

6. *Appendix: bibliographic notes*

Construction of self-adjoint extensions prior to [Friedrichs 1934] was non-constructive, and non-canonical: [Neumann 1929], [Stone 1929], [Stone 1930]. Friedrichs' construction was not *overtly* about what are now called Sobolev spaces, but used a bit of that idea, a few years before [Sobolev 1937] or [Sobolev 1938].

[Borel 1997] refers to [Reed-Simon 1980] for what would be commonly called *functional analysis* facts, in particular for the essential self-adjointness criterion above, that the adjoint have no non-real eigenvalues. Appendices in [Lang 1975] treat similar issues succinctly.

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