Bernstein's analytic continuation of complex powers

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- Analytic continuation of distributions
- Statement of the theorems on analytic continuation
- Bernstein's proofs
- Proof of the Lemma: the Bernstein polynomial
- Proof of the Proposition: estimates on zeros

Let f be a polynomial in $x_1, \ldots, x_n$ with real coefficients. For complex s, let f_+^s be the function defined by

$$f_+^s(x) = f(x)^s \quad \text{if } f(x) \geq 0$$

$$f_+^s(x) = 0 \quad \text{if } f(x) \leq 0$$

Certainly for $\Re(s) \geq 0$ the function f_+^s is *locally integrable*. For s in this range, we can defined a *distribution*, denoted by the same symbol f_+^s, by

$$f_+^s(\phi) := \int_{\mathbf{R}^n} f_+^s(x)\, \phi(x)\, dx$$

where ϕ is in $C_c^\infty(\mathbf{R}^n)$, the space of compactly-supported smooth real-valued functions on $\mathbf{R}^n$.

The object is to analytically continue the distribution f_+^s, as a meromorphic (distribution-valued) function of s. This type of question was considered in several provocative examples in I.M. Gelfand and G.E. Shilov's *Generalized Functions, volume I.* (One should also ask about analytic continuation as a *tempered* distribution). In a lecture at the 1963 Amsterdam Congress, I.M. Gelfand refined this question to require further that one show that the 'poles' lie in a finite number of arithmetic progressions.

Bernstein proved the result in 1967, under a certain 'regularity' hypothesis on the zero-set of the polynomial f. (Published in *Journal of Functional Analysis and Its Applications, 1968*, translated from Russian).

The present discussion includes some background material from complex function theory and from the theory of distributions.

1 Analytic continuation of distributions

First we recall the nature of the topologies on test functions and on distributions. Let $C_c^\infty(U)$ be the collection of compactly-supported smooth functions with support inside a set $U \subset \mathbf{R}^n$. As usual, for U *compact*, we have a countable family of seminorms on $C_c^\infty(U)$:

$$\mu_\nu(f) := \sup_x |D^\nu f|$$

where for $\nu = (\nu_1, \ldots, \nu_n)$ we write, as usual,

$$D^\nu = \left(\frac{\partial}{\partial x_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\nu_n}$$

It is elementary to show that (for U compact) $C_c^\infty(U)$ is a complete, locally convex topological space (a Frechet space). To treat U not necessarily compact (e.g., $\mathbf{R}^n$ itself) let

$$U_1 \subset U_2 \subset \ldots$$

be compact subsets of U so that their union is all of U. Then $C_c^\infty(U)$ is the union of the spaces $C_c^\infty(U_i)$, and we give it *with the locally convex direct limit topology.*

The spaces $\mathcal{D}^*(U)$ and $\mathcal{D}^*(\mathbf{R}^n)$ of **distributions** on U and on $\mathbf{R}^n$, respectively, are the continuous duals of $\mathcal{D}(U) = C_c^\infty(U)$ and $\mathcal{D}(\mathbf{R}^n) = C_c^\infty(\mathbf{R}^n)$. For present purposes, the topology we put on the continuous dual space V^* of a topological vectorspace V is the *weak-$*$ topology:* a sub-basis near 0 in V^* is given by sets

$$U_{v,\epsilon} := \{\lambda \in V^* : |\lambda(v)| < \epsilon\}$$

In this context, a V^*-valued function f on an open subset Ω of $\mathbf{C}$ is **holomorphic** on Ω if, for every $v \in V$, the $\mathbf{C}$-valued function

$$z \to f(z)(v)$$

on Ω is holomorphic in the usual sense. This notion of holomorphy might be more pedantically termed '*weak-$*$ holomorphy*', since reference to the topology might be required.

If $z_o \in \Omega$ for an open subset Ω of $\mathbf{C}$ and f is a holomorphic V^*-valued function on $\Omega - z_o$, say that f is **weakly meromorphic** at z_o if, for every $v \in V$, the $\mathbf{C}$-valued function $z \to f(z)(v)$ has a *pole* (as opposed to essential singularity) at z_o. Say that f is **strongly meromorphic** at z_o if the orders of these poles are *bounded independently of* v. That is, f is strongly meromorphic at z_o if there is an integer n and an open set Ω containing z_o so that, for all $v \in V$ the $\mathbf{C}$-valued function

$$z \to (z - z_o)^n f(z)(v)$$

is holomorphic on Ω. If n is the least integer f so that $(z-z_o)^n f$ is holomorphic at z_o, then f *is of order* $-n$ *at* z_o, etc.

To say that f is **strongly meromorphic** on an open set Ω is to require that there be a set S of points of Ω *with no accumulation point in* Ω so that f is holomorphic on $\Omega - S$, and so that f is strongly meromorphic at each point of S.

For brevity, but risking some confusion, we will often say 'meromorphic' instead of 'strongly meromorphic'.

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2 Statement of the theorems on analytic continuation

Let $\mathcal{O}$ be the polynomial ring $\mathbf{R}[x_1, \ldots, x_n]$. For $z \in \mathbf{R}^n$, let $\mathcal{O}_z$ be the *local ring* at z, i.e., the ring of ratios P/Q of polynomials where the denominator does not vanish at z. Let $\mathbf{m}_z$ be the maximal ideal of $\mathcal{O}_z$ consisting of elements of $\mathcal{O}_z$ whose numerator vanishes at z. Let I_z (depending upon f) be the ideal in $\mathcal{O}_z$ generated by

$$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$$

A point $z \in \mathbf{R}^n$ is **simple** with respect to the polynomial f if

- $f(z) = 0$
- for some N we have $I_z \supset \mathbf{m}_z^N$
- There are $\alpha_i \in \mathbf{m}_z$ so that $f = \sum_i \alpha_i \frac{\partial f}{\partial x_i}$

Remarks: The second condition is equivalent to the assertion that $\mathcal{O}_z/I_z$ is finite-dimensional. The simplest situation in which the second condition holds is when $I_z = \mathcal{O}_z$, i.e., some partial derivative of f is non-zero at z. The third condition does not follow from the first two. For example, Bernstein points out that with

$$f(x,y) = x^5 + y^5 + x^2y^2$$

the first two conditions hold but the third does not.

Theorem (local version): If z is a simple point with respect to f, then there is a neighborhood U of x so that the distribution

$$f^s_{+,U}(\phi) := \int f^s_+(x)\, \phi(x)\, dx$$

on test functions $\phi \in C_c^\infty(U)$ *on* U has an analytic continuation to a meromorphic element in the continuous dual of $C_c^\infty(U)$.

Theorem (global version): If *all* real zeros of $f(x)$ are *simple* (with respect to f), then f_+^s is a meromorphic (distribution-valued) function of $s \in \mathbf{C}$.

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3 Bernstein's proof

Let R_z be the ring of linear differential operators with coefficients in $\mathcal{O}_z$. Note that R_z is both a left and a right $\mathcal{O}$-module: for $D \in R_z$, for $f, g \in \mathcal{O}$ and ϕ a smooth function near z, the definition is

$$(fDg)(\phi) := f\,D(g\phi)$$

Lemma: There is a differential operator $D \in R_z$ and a non-zero *'Bernstein polynomial'* H in a single variable so that

$$D(f^{n+1}) = H(n)f^n$$

for any natural number n. (*Proof below.*)

Proof of Local Theorem from Lemma: Let U be a small-enough neighborhood of z so that on it all coefficients of D are holomorphic on U. For sufficiently large natural numbers n the function f_+^{n+1} is continuously differentiable, so we have

$$Df_+^{n+1} = H(n)f_+^n$$

For each fixed $\phi \in C_c^\infty(U)$ consider the function

$$g(s) := (Df_+^{s+1} - H(s)f_+^s)(\phi)$$

The hypotheses of the proposition below are satisfied, so the equality for all large-enough natural numbers implies equality everywhere:

$$Df_+^{s+1} = H(s)f_+^s$$

This gives us

$$f_+^s = \frac{Df_+^{s+1}}{H(s)}$$

Now we claim that for any $0 \le n \in \mathbf{Z}$ the distribution f_+^s on $C_c^\infty(U)$ is meromorphic for $\Re(s) > -n$. For $n = 0$ this is certain. The formula just derived then gives the induction step. Further, this argument makes clear that the 'poles' of f_+^s *restricted to* $C_c^\infty(U)$ are concentrated on the finite collection of arithmetic progressions

$$\lambda_i, \lambda_i - 1, \lambda_i - 2, \ldots$$

where the λ_i are the roots of $H(s)$. In particular, the *order* of the pole of f_+^s at a point s_o is equal to the number of roots λ_i so that s_o lies among

$$\lambda_i, \lambda_i - 1, \lambda_i - 2, \ldots$$

In particular, the distribution f_+^s really is (strongly) meromorphic. This proves the Theorem, granting the Lemma and granting the Proposition. ♣

Proposition (attributed by Bernstein to 'Carlson'): If g is an analytic function for $\Re s > 0$ and $|g(s)| < be^{c\Re s}$ and if $g(n) = 0$ for all sufficiently large natural numbers n, then $g \equiv 0$.

Proof of Global Theorem: Invoking the Local Theorem *and its proof above*, for each $z \in \mathbf{R}^n$ we choose a neighborhood U_z of z in which f_+^s is meromorphic, so that U_z is *Zariski-open*, i.e., is the complement of a finite union of zero sets of polynomials. Indeed, writing

$$f(x) = \sum \alpha_i \frac{\partial f}{\partial x_i}$$

with $\alpha_i \in \mathcal{O}_z$, as in the proof of the Local Theorem, let $\alpha_i = g_i/h_i$ with polynomials g_i and h_i, and take U_z to be the complement of the union of the zero-sets of the denominators h_i.

Then Hilbert's Basis Theorem implies that the whole $\mathbf{R}^n$ is covered by finitely-many $U_{z_1}, \ldots, U_{z_n}$ of these Zariski-opens. Then make a partition of unity subordinate to this finite cover, i.e., take $\psi_1, \ldots, \psi_n$ so that $\psi_i \geq 0$, $\sum \psi_i \equiv 1$, and $\mathrm{spt}\psi_i \subset U_{z_i}$. Then

$$f_+^s = \sum_i \psi_i f_+^s$$

By choice of the neighborhoods U_{z_i}, the right-hand side is a finite sum of meromorphic (distribution-valued) functions.

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4 Proof of the Lemma: the Bernstein polynomial

Now we prove existence of the differential operator D and the 'Bernstein polynomial' H. This is the most serious part of this proof. (The complex function theory proposition is not entirely trivial, but is approximately standard).

Proof of Lemma: Let

$$P := \sum \alpha_i \frac{\partial}{\partial x_i} \in R_z$$

where the $\alpha_i \in \mathbf{m}_z$ are so that

$$f = \sum \alpha_i \frac{\partial f}{\partial x_i} \in R_z$$

Also put

$$S_i := \frac{\partial f}{\partial x_i} P - f \frac{\partial}{\partial x_i}$$
$$= \frac{\partial f}{\partial x_i}(P+1) - \frac{\partial}{\partial x_i} Q$$

Then we have

$$P(f) = \sum_i \alpha_i \frac{\partial f}{\partial x_i} = f$$

$$S_i f = \frac{\partial f}{\partial x_i} f - f \frac{\partial f}{\partial x_i} = 0$$

Thus, by Leibniz' formula,

$$P(f^n) = n f^n \quad S_i(f^n) = 0$$

Sublemma: There is a non-zero polynomial M in one variable so that $M(P)$ can be written in the form

$$M(P) = \sum_i J_i \frac{\partial f}{\partial x_i}$$

for some $J_i \in R_z$.

Proof of Sublemma: Write, as usual,

$$|\nu| = \nu_1 + \ldots + \nu_n$$

For a natural number m, write

$$P^m = \sum_{|\nu| \leq m} D^\nu \gamma_{m,\nu}$$

where $\gamma_{m,\nu} \in \mathcal{O}_z$. That is, we move all the coefficients to the right of the differential operators. That this is possible is easy to see: for example,

$$x_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} x_i$$

is 0 or -1 as $i = j$ or not.

Further, the coefficients $\gamma_{m,\nu}$ are *polynomials* in the α_i. Thus,

$$\gamma_{m,\nu} \in \mathbf{m}_z^{|\nu|}$$

Then taking $M(P)$ of the form

$$M(P) = \sum_{m \leq q} b_m P^m = \sum_{m,\nu} D^\nu b_m \gamma_{m,\nu}$$

with $b_m \in \mathbf{R}$, the condition of the sublemma will be met if

$$\sum_m b_m \gamma_{m,\nu} \in I_z$$

for all indices ν. If $|\nu| \geq N$, where $I \supset \mathbf{m}^N$, then this condition is automatically fulfilled. Thus, there are finitely-many conditions

$$\sum_m b_m \bar{\gamma}_{m,\nu} = 0$$

where $\bar{\gamma}_{m,\nu}$ is the image of $\gamma_{m,\nu}$ in $\mathcal{O}_z/\mathbf{m}_z^N$. Since the latter quotient is, by hypothesis, a finite-dimensional vector space, the collection of such conditions gives a finite collection of homogeneous equations in the coefficients b_m. More specifically, there are

$$\dim \mathcal{O}_z/\mathbf{m}_z^N \times \mathrm{card}\{\nu : |\nu| < N\}$$

such conditions. By taking q large enough we assure the existence of a non-trivial solution $\{b_m\}$. This proves the Sublemma. ♣

Returning to the proof of the Lemma: as an equation in R_z

$$M(P)(P+1) = \sum J_i \frac{\partial}{\partial x_i}(P+1) = \sum J_i S_i + \sum J_i \frac{\partial}{\partial x_i} f$$

Now put

$$D = \sum J_i \frac{\partial}{\partial x_i}$$

$$H(P) = M(P)(P+1)$$

Then we have

$$D(f^{n+1}) = (\sum J_i \frac{\partial}{\partial x_i} f)(f^n) =$$

$$= H(P)(f^n) = H(n) f^n$$

as desired. This proves the Lemma, constructing the differential operator D. ♣

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5 Proof of the Proposition: estimates on zeros

The result we need is a standard one from complex function theory, although it is not so elementary as to be an immediate corollary of Cauchy's Theorem:

Proposition: If g is an analytic function for $\Re s > 0$ and $|g(s)| < be^{c\Re s}$ and if $g(n) = 0$ for all sufficiently large natural numbers n, then $g \equiv 0$.

Proof of Proposition: Consider

$$G(z) := e^{-c}\, g(\frac{z+1}{z-1})$$

Then g is turned into a bounded function G on the disc, with zeros at points $(n-1)/(n+1)$ for sufficiently large natural numbers n.

We claim that, *for a bounded function G on the unit disc with zeros ρ_i, either $G \equiv 0$ or*

$$\sum_i (1 - |\rho_i|) < +\infty$$

If we prove this, then in the situation at hand the natural numbers are mapped to

$$\rho_n := (n-1)/(n+1) = 1 - \frac{1}{n+1}$$

so here

$$\sum_n (1 - |\rho_n|) = \sum_n \frac{1}{n+1} = +\infty$$

Thus, we would conclude $G \equiv 0$ as desired.

We recall *Jensen's formula:* for any holomorphic function G on the unit disc with $G(0) \neq 0$ and with zeros $\rho_1, \ldots$, for $0 < r < 1$ we have

$$|G(0)| \, \Pi_{|\rho_i| \le r} \frac{r}{|\rho_i|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |G(re^{i\theta})| \, d\theta \right\}$$

Granting this, our assumed boundedness of G on the disc gives us an absolute constant C so that for all N

$$|G(0)| \, \Pi_{|\rho_i| \le r} \frac{r}{|\rho_i|} \le C$$

(We can harmlessly divide by a suitable power of z to guarantee that $G(0) \neq 0$.) Then, letting $r \to 1$,

$$\Pi \, |\rho_i| \le |G(0)|^{-1} C^{-1}$$

For an infinite product of positive real numbers $|\rho_i|$ less than 1 to have a value > 0, it is elementary that we must have

$$\sum_i (1 - |\rho_i|) < +\infty$$

as claimed. This proves the proposition. ♣

While we're here, let's recall the proof of Jensen's formula (e.g., as in Rudin's *Real and Complex Analysis*, page 308). Fix $0 < r < 1$ and let

$$H(z) := G(z) \, \Pi \, \frac{r^2 - \bar{\rho} z}{r(\rho - z)} \, \Pi \, \frac{\rho}{\rho - z}$$

where the first product is over roots ρ with $|\rho| < r$ and the second is over roots with $|\rho| = r$. Then H is holomorphic and non-zero in an open disk of radius

$r+\epsilon$ for some $\epsilon > 0$. Thus, $\log|H|$ is *harmonic* in this disk, and we have the *mean value property*

$$\log|H(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|H(re^{i\theta})|\,d\theta$$

On one hand,

$$|H(0)| = |G(0)| \Pi \frac{r}{|\rho|}$$

On the other hand, if $|z| = r$ the factors

$$\frac{r^2 - \bar{\rho}z}{r(\rho - z)}$$

have absolute value 1. Thus,

$$\log|H(re^{i\theta})| = \log|G(re^{i\theta})| - \sum_{|\rho|=r} \log|1 - e^{i(\theta - \arg\rho)}|$$

where

$$e^{i\arg\rho} = \rho$$

As noted in Rudin (see below),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|1 - e^{i\theta}|\,d\theta = 0$$

Therefore, the integral appearing in the assertion of the mean value property is unchanged upon replacing H by G. Putting this all together gives Jensen's formula. ♣

And let's do the integral computation, following Rudin. There is a function $\lambda(z)$ on the open unit disc so that

$$\exp(\lambda(z)) = 1 - z$$

since the disc is simply-connected. We uniquely specify this λ by requiring that $\lambda(0) = 0$. We have

$$\Re\lambda(z) = \log|1 - z| \text{ and } |\Im\lambda(z)| < \frac{\pi}{2}$$

Let $\delta > 0$ be small. Let $\Gamma = \Gamma_\delta$ be the path which goes (counterclockwise) around the unit circle from $e^{i\delta}$ to $e^{(2\pi-\delta)i}$ and let $\gamma = \gamma_\delta$ be the path which goes (clockwise) around a small circle centered at 1 from $e^{(2\pi-\delta)i}$ to $e^{i\delta}$. Then

$$\frac{1}{2\pi}\int_{0}^{2\pi} \log|1 - e^{i\theta}|\,d\theta = \lim_{\delta\to 0} \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} \log|1 - e^{i\theta}|\,d\theta =$$

$$= \Re \left[\frac{1}{2\pi i} \int_\Gamma \lambda(z) \, \frac{dz}{z} \right] =$$
$$= \Re \left[\frac{1}{2\pi i} \int_\gamma \lambda(z) \, \frac{dz}{z} \right]$$

by Cauchy's theorem.

Elementary estimates show that the latter integral has a bound of the form

$$C \, \delta \, \log(1/\delta)$$

which goes to 0 as $\delta \to 0$.