(*March 1, 2012*)

Compact operators, Hilbert-Schmidt operators

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

---

1. *Spectral theorem for self-adjoint compact operators*

The following slightly clever rewrite of the operator norm is a substantial part of the existence proof for eigenvectors and eigenvalues.

[1.0.1] Proposition: A continuous *self-adjoint* operator T on a Hilbert space V has operator norm $|T| = \sup_{|v|\le 1} |Tv|$ expressible as

$$|T| \;=\; \sup_{|v|\le 1} |\langle Tv, v\rangle|$$

Proof: On one hand, certainly $|\langle Tv, v\rangle| \le |Tv|\cdot|v|$ by Cauchy-Schwarz-Bunyakowsky, giving the easy direction of inequality.

On the other hand, let $\sigma = \sup_{|v|\le 1} |\langle Tv, v\rangle|$. A polarization identity gives

$$2\langle Tv, w\rangle + 2\langle Tw, v\rangle \;=\; \langle T(v+w), v+w\rangle - \langle T(v-w), v-w\rangle$$

With $w = t\cdot Tv$ with $t>0$, since $T=T^*$, $\langle Tv, w\rangle = \langle Tw, v\rangle$ is non-negative real. Taking absolute values,

$$4\langle Tv, t\cdot Tv\rangle \;=\; \Big|\langle T(v+t\cdot Tv), v+t\cdot Tv\rangle - \langle T(v-t\cdot Tv), v-t\cdot Tv\rangle\Big|$$

$$\le\; \Big|\langle T(v+t\cdot Tv), v+t\cdot Tv\rangle\Big| + \Big|\langle T(v-t\cdot Tv), v-t\cdot Tv\rangle\Big|$$

Using $|\langle Tx, x\rangle| \le \sigma$ with $x = v \pm tTv$,

$$|\langle T(v\pm tTv), v\pm tTv\rangle| \;\le\; \sigma\cdot\langle v\pm tTv, v\pm tTv\rangle \;=\; \sigma\cdot|v\pm tTv|^2$$

Combining this with the previous gives the bound on $4\langle Tv, t\cdot Tv\rangle$:

$$4\langle Tv, t\cdot Tv\rangle \;\le\; \sigma\cdot|v+t\cdot Tv|^2 + \sigma\cdot|v-t\cdot Tv|^2 \quad=\quad 2\sigma\cdot\big(|v|^2 + t^2\cdot|Tv|^2\big)$$

Divide through by $4t$ and set $t = |v|/|Tv|$ to minimize the right-hand side, obtaining

$$|Tv|^2 \;\le\; \sigma\cdot|v|\cdot|Tv|$$

giving the other inequality. ///

Recall that a continuous linear operator $T: V\to V$ is *compact* when the image of the unit ball by T has compact closure.

[1.0.2] Theorem: A compact self-adjoint operator T has largest eigenvalue $\pm|T|$, and an orthonormal basis of eigenfunctions. The number of eigenvalues λ larger than a given constant $c>0$ is finite. In particular, multiplicities are finite.

Proof: The crucial point is existence of eigenvalue $\pm|T|$.

Suppose $|T| > 0$. Using the re-characterization of operator norm, let v_i be a sequence of unit vectors such that $|\langle Tv_i, v_i\rangle| \to |T|$. Take a sign and replace v_i by a subsequence so that $\langle Tv_i, v_i\rangle \to \pm|T|$. Let λ be the corresponding $\pm|T|$.

On one hand, using $\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$,

$$0 \;\le\; |Tv_i - \lambda v_i|^2 \;=\; |Tv_i|^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2|v_i|^2 \;\le\; \lambda^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2$$

By assumption, the right-hand side goes to 0. Using compactness, replace v_i with a subsequence such that Tv_i has limit w. Then the inequality shows that $\lambda v_i \to w$, so $v_i \to \lambda^{-1}w$. Thus, by continuity of T, $Tw = \lambda w$.

The other assertions follow by the short natural arguments:

If $|T| = 0$, then $T = 0$. Otherwise, T has a non-zero eigenvalue. The orthogonal complement of the span of all eigenvectors with non-zero values is T-stable: for $w \perp v$,

$$\langle v, Tw\rangle = \langle Tv, w\rangle = \lambda\langle v, w\rangle = 0 \qquad (\text{for } Tv = \lambda v \text{ and } \langle v, w\rangle = 0)$$

The restriction of T to that orthogonal complement is still compact (!), so unless that restriction is 0, T has a non-zero eigenvalue there, too. Contradiction.

For $\lambda \neq 0$, the λ-eigenspace being infinite-dimensional would contradict the compactness of T: the unit ball in an infinite-dimensional inner-product space is not compact, as any infinite orthonormal set is a sequence with no convergent subsequence. Similarly, for $c > 0$, the set of eigenvalues (counting multiplicities) larger than c being infinite would contradict compactness. Thus, 0 is the only limit-point of eigenvalues. ///

---

2. *Hilbert-Schmidt operators are compact*

Let X, Y be locally compact, Hausdorff, countably-based topological spaces with nice measures.

Let $K(x, y) \in C_c^o(X \times Y)$, and define $T : L^2(Y) \to L^2(X)$ by the *integral operator*

$$Tf(x) \;=\; \int_Y K(x, y)\, f(y)\, dy$$

[2.0.1] Theorem: The operator T is *compact.*

Proof: We show that T is an operator-norm limit of *finite-rank* operators. [1] The following section reviews compactness of operator-norm limits of finite-rank operators.

Given $\varepsilon > 0$, find a *finite* collection of functions f_i, F_i such that

$$\sup_{x,y} \Big| K(x, y) - \sum_i f_i \otimes F_i \Big| \;<\; \varepsilon$$

[1] A *finite-rank* operator is one that has finite-dimensional image. The usual convention is that finite-rank operators are *also* understood to be *continuous.* This continuity does not follow from finite-dimensionality of the image: on infinite-dimensional Hilbert-spaces V there are linear maps $V \to \mathbb{C}$ that are *not* continuous, although constructions of these require an equivalent of Axiom of Choice. In any case, our finite-rank operators are assumed continuous.

Fix $\varepsilon > 0$. For each (x,y) in the support of K, by continuity of K, let $U_x \times V_y$ be a neighborhood of (x,y) such that $|K(x,y) - K(x',y')| < \varepsilon$ for $x' \in U_x$ and $y' \in V_y$, where U_x and V_y are neighborhoods of x, y, respectively.

By compactness of the support of $K(x,y)$, there are finitely-many x_j, y_j such that $U_j \times V_j$ (abbreviating $U_{x_j} \times V_{y_j}$) cover the support of $K(x,y)$. Let

$$\varphi_j \;=\; \text{char fcn } U_j \qquad\text{and}\qquad \Phi_j \;=\; K(x_j, y_j)\cdot(\text{char fcn } V_j)$$

The sets $U_j \times V_j$ *overlap*, so $K \neq \sum_j \varphi_j \otimes \Phi_j$, necessitating minor adjustments.

One way to compensate for the overlaps is by subtracting two-fold overlaps, adding back three-fold overlaps, subtracting four-fold, and so on: let

$$Q \;=\; \sum_i \varphi_i \otimes \Phi_i - \sum_{i_1<i_2} \min(\varphi_{i_1}, \varphi_{i_2}) \otimes \min(\Phi_{i_1}, \Phi_{i_2}) + \sum_{i_1<i_2<i_3} \min(\varphi_{i_1}, \varphi_{i_2}, \varphi_{i_3}) \otimes \min(\Phi_{i_1}, \Phi_{i_2}, \Phi_{i_3}) - \ldots$$

Because the subcover is finite, Q is a finite linear combination $Q = \sum_j f_j \otimes F_j$. By construction, $\sup_{x,y} \big|K(x,y) - Q(x,y\big| < \varepsilon$. The operator

$$f \;\longrightarrow\; \int_Y Q(x,y)\, f(y)\; dy$$

is finite-rank, because the image is in the span of the finitely-many f_i appearing in the definition of $Q(x,y)$.

Let χ be the characteristic function of the closure $\overline{U}$ of a compact-closure open U containing the support of K. For every $\varepsilon > 0$, the opens U_x and U_y can be chosen inside U. Then

$$\Big|\int_Y Q(x,y)\, f(y)\; dy \;-\; \int_Y K(x,y)\, f(y)\; dy\Big| \;\le\; \int_Y |Q(x,y) - K(x,y)|\cdot|f(y)|\; dy$$

$$<\; \varepsilon \int_Y |\chi(x,y)|\cdot|f(y)|\; dy \;\le\; \varepsilon\cdot|\chi|_{L^2}\cdot|f|_{L^2}$$

Thus, the operator norm of the difference can be made arbitrarily small, proving that the operator T given by $K(x,y) \in C_c^o(X \times Y)$ is an operator-norm limit of finite-rank operators, so compact. ///

---

3. *Operator-norm limits of finite-rank operators are compact*

We recall the argument that *operator-norm limits of finite-rank operators are compact*, and conversely. [2]

Proof: Let $T = \lim_i T_i$, where $T_i : X \to Y$ is finite-rank from Hilbert space X to Hilbert space Y. Let B be the unit ball in X. We show that TB has compact closure by showing that it is *totally bounded*, that is, for every $\varepsilon > 0$ it can be covered by finitely-many ε-balls.

Given $\varepsilon > 0$, let i be large-enough so that $|T - T_i| < \varepsilon$. Since T_i is finite-rank, $T_i B$ is covered by finitely-many ε-balls $B_1, \ldots, B_n$ in Y with respective centers $y_1, \ldots, y_n$. For $x \in B$, with $T_i x \in B_j$,

$$|Tx - y_j| \;\le\; |Tx - T_i x| + |T_i x - y_j| \;<\; \varepsilon + \varepsilon$$

Thus, TB is covered by a finite number of 2ε-balls. This holds for every $\varepsilon > 0$, so TB is *totally bounded*. ///

[2] In Banach spaces, the converse is false: there are compact operators which are *not* operator-norm limits of finite-rank operators. The counter-examples are difficult, due to Per Enflo.

Recall the proof that *total boundedness* of a set E in a complete metric space implies compact closure:

Since metric spaces have countable local bases, it suffices to show *sequential* compactness. That is, a sequence $\{v_i\}$ in E, exhibit a convergent subsequence.

Cover E by finitely-many 2^{-1}-balls, choose one, call it B_1, with infinitely-many v_i in $E \cap B_1$, and let w_1 be one of those infinitely-many v_i.

Next, cover E by finitely-many 2^{-2}-balls. Certainly $E \cap B_1$ is covered by these, and $E \cap B_1 \cap B_2$ contains infinitely-many v_i for at least one of these, call it B_2. Let $w_2 \in E \cap B_1 \cap B_2$ be one of these v_i, other than w_1.

Inductively, find an infinite subsequence w_n of distinct points, with $w_n \in E \cap B_1 \cap \ldots \cap B_n$, where B_n is of radius 2^{-n}. The sequence w_i is Cauchy. ///
