1. **Spectral theorem for self-adjoint compact operators**

The following slightly clever rewrite of the operator norm is a substantial part of the existence proof for eigenvectors and eigenvalues.

**[1.0.1] Proposition:** A continuous self-adjoint operator $T$ on a Hilbert space $V$ has operator norm $|T| = \sup_{|v| \leq 1} |(Tv, v)|$ expressible as $|T| = \sup_{|v| \leq 1} |Tv|$.

**Proof:** On one hand, certainly $|(Tv, v)| \leq |Tv| \cdot |v|$ by Cauchy-Schwarz-Bunyakowsky, giving the easy direction of inequality.

On the other hand, let $\sigma = \sup_{|v| \leq 1} |(Tv, v)|$. A polarization identity gives

$$2\langle Tv, w \rangle + 2\langle Tw, v \rangle = \langle T(v + w), v + w \rangle - \langle T(v - w), v - w \rangle$$

With $w = t \cdot Tv$ with $t > 0$, since $T = T^*$, $\langle Tw, v \rangle = \langle Tv, v \rangle$ is non-negative real. Taking absolute values,

$$4\langle Tv, t \cdot Tv \rangle = |\langle T(v + t \cdot Tv), v + t \cdot Tv \rangle - \langle T(v - t \cdot Tv), v - t \cdot Tv \rangle|$$

$$\leq |\langle T(v + t \cdot Tv), v + t \cdot Tv \rangle| + |\langle T(v - t \cdot Tv), v - t \cdot Tv \rangle|$$

Using $|\langle Tx, x \rangle| \leq \sigma$ with $x = v \pm tTv$,

$$|\langle T(v \pm tTv), v \pm tTv \rangle| \leq \sigma \cdot |v \pm tTv, v \pm tTv| = \sigma \cdot |v \pm tTv|^2$$

Combining this with the previous gives the bound on $4\langle Tv, t \cdot Tv \rangle$:

$$4\langle Tv, t \cdot Tv \rangle \leq \sigma \cdot |v + t \cdot Tv|^2 + \sigma \cdot |v - t \cdot Tv|^2 = 2\sigma \cdot (|v|^2 + t^2 \cdot |Tv|^2)$$

Divide through by $4t$ and set $t = |v|/|Tv|$ to minimize the right-hand side, obtaining

$$|Tv|^2 \leq \sigma \cdot |v| \cdot |Tv|$$

giving the other inequality. ///

Recall that a continuous linear operator $T : V \to V$ is *compact* when the image of the unit ball by $T$ has compact closure.

**[1.0.2] Theorem:** A compact self-adjoint operator $T$ has largest eigenvalue $\pm |T|$, and an orthonormal basis of eigenfunctions. The number of eigenvalues $\lambda$ larger than a given constant $c > 0$ is finite. In particular, multiplicities are finite.
**Proof:** The crucial point is existence of eigenvalue $\pm|T|$. Suppose $|T| > 0$. Using the re-characterization of operator norm, let $v_i$ be a sequence of unit vectors such that $|\langle Tv_i, v_i \rangle| \to |T|$. Take a sign and replace $v_i$ by a subsequence so that $\langle Tv_i, v_i \rangle \to \pm|T|$. Let $\lambda$ be the corresponding $\pm|T|$. On one hand, using $\langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, Tv \rangle$, 

$$0 \leq |Tv_i - \lambda v_i|^2 = |Tv_i|^2 - 2\lambda\langle Tv_i, v_i \rangle + \lambda^2 |v_i|^2 \leq \lambda^2 - 2\lambda\langle Tv_i, v_i \rangle + \lambda^2$$

By assumption, the right-hand side goes to 0. Using compactness, replace $v_i$ with a subsequence such that $Tv_i$ has limit $w$. Then the inequality shows that $\lambda v_i \to w$, so $v_i \to \lambda^{-1}w$. Thus, by continuity of $T$, $Tw = \lambda w$.

The other assertions follow by the short natural arguments:

If $|T| = 0$, then $T = 0$. Otherwise, $T$ has a non-zero eigenvalue. The orthogonal complement of the span of all eigenvectors with non-zero values is $T$-stable: for $w \perp v$, 

$$\langle v, Tw \rangle = \langleTv,w\rangle = \lambda\langle v,w \rangle = 0 \quad \text{ (for } T v = \lambda v \text{ and } \langle v,w \rangle = 0)$$

The restriction of $T$ to that orthogonal complement is still compact (!), so unless that restriction is 0, $T$ has a non-zero eigenvalue there, too. Contradiction.

For $\lambda \neq 0$, the $\lambda$-eigenspace being infinite-dimensional would contradict the compactness of $T$: the unit ball in an infinite-dimensional inner-product space is not compact, as any infinite orthonormal set is a sequence with no convergent subsequence. Similarly, for $c > 0$, the set of eigenvalues (counting multiplicities) larger than $c$ being infinite would contradict compactness. Thus, 0 is the only limit-point of eigenvalues.  

2. **Hilbert-Schmidt operators are compact**

Let $X,Y$ be locally compact, Hausdorff, countably-based topological spaces with nice measures.

Let $K(x,y) \in C^0_c(X \times Y)$, and define $T : L^2(Y) \to L^2(X)$ by the integral operator

$$Tf(x) = \int_Y K(x,y) f(y) \, dy$$

**[2.0.1] Theorem:** The operator $T$ is compact.

**Proof:** We show that $T$ is an operator-norm limit of finite-rank operators. The following section reviews compactness of operator-norm limits of finite-rank operators.

Given $\varepsilon > 0$, find a finite collection of functions $f_i, F_i$ such that

$$\sup_{x,y} \left| K(x,y) - \sum_i f_i \otimes F_i \right| < \varepsilon$$

[1] A finite-rank operator is one that has finite-dimensional image. The usual convention is that finite-rank operators are also understood to be continuous. This continuity does not follow from finite-dimensionality of the image: on infinite-dimensional Hilbert-spaces $V$ there are linear maps $V \to \mathbb{C}$ that are not continuous, although constructions of these require an equivalent of Axiom of Choice. In any case, our finite-rank operators are assumed continuous.
Fix $\varepsilon > 0$. For each $(x, y)$ in the support of $K$, by continuity of $K$, let $U_x \times V_y$ be a neighborhood of $(x, y)$ such that $|K(x, y) - K(x', y')| < \varepsilon$ for $x' \in U_x$ and $y' \in V_y$, where $U_x$ and $V_y$ are neighborhoods of $x, y$, respectively.

By compactness of the support of $K(x, y)$, there are finitely-many $x_j, y_j$ such that $U_j \times V_j$ (abbreviating $U_{x_j} \times V_{y_j}$) cover the support of $K(x, y)$. Let

$$\varphi_j = \text{char fcn } U_j \quad \text{and} \quad \Phi_j = K(x_j, y_j) \cdot (\text{char fcn } V_j)$$

The sets $U_j \times V_j$ overlap, so $K \neq \sum_j \varphi_j \otimes \Phi_j$, necessitating minor adjustments.

One way to compensate for the overlaps is by subtracting two-fold overlaps, adding back three-fold overlaps, subtracting four-fold, and so on: let

$$Q = \sum_i \varphi_i \otimes \Phi_i - \sum_{i_1 < i_2} \min (\varphi_{i_1}, \varphi_{i_2}) \otimes \min (\Phi_{i_1}, \Phi_{i_2}) + \sum_{i_1 < i_2 < i_3} \min (\varphi_{i_1}, \varphi_{i_2}, \varphi_{i_3}) \otimes \min (\Phi_{i_1}, \Phi_{i_2}, \Phi_{i_3}) - \ldots$$

Because the subcover is finite, $Q$ is a finite linear combination $Q = \sum_j f_j \otimes F_j$. By construction, $\sup_{x,y} |K(x, y) - Q(x, y)| < \varepsilon$. The operator

$$f \rightarrow \int_Y Q(x, y) f(y) \, dy$$

is finite-rank, because the image is in the span of the finitely-many $f_i$ appearing in the definition of $Q(x, y)$.

Let $\chi$ be the characteristic function of the closure $\overline{U}$ of a compact-closure open $U$ containing the support of $K$. For every $\varepsilon > 0$, the opens $U_x$ and $U_y$ can be chosen inside $U$. Then

$$\left| \int_Y Q(x, y) f(y) \, dy - \int_Y K(x, y) f(y) \, dy \right| \leq \int_Y |Q(x, y) - K(x, y)| \cdot |f(y)| \, dy$$

$$< \varepsilon \int_Y |\chi(x, y)| \cdot |f(y)| \, dy \leq \varepsilon \cdot |\chi|_{L^2} \cdot |f|_{L^2}$$

Thus, the operator norm of the difference can be made arbitrarily small, proving that the operator $T$ given by $K(x, y) \in C_0^c(X \times Y)$ is an operator-norm limit of finite-rank operators, so compact.

3. Operator-norm limits of finite-rank operators are compact

We recall the argument that operator-norm limits of finite-rank operators are compact, and conversely. [2]

Proof: Let $T = \lim \limits_{i \rightarrow \infty} T_i$, where $T_i : X \rightarrow Y$ is finite-rank from Hilbert space $X$ to Hilbert space $Y$. Let $B$ be the unit ball in $X$. We show that $TB$ has compact closure by showing that it is totally bounded, that is, for every $\varepsilon > 0$ it can be covered by finitely-many $\varepsilon$-balls.

Given $\varepsilon > 0$, let $i$ be large-enough so that $|T - T_i| < \varepsilon$. Since $T_i$ is finite-rank, $T_i B$ is covered by finitely-many $\varepsilon$-balls $B_1, \ldots, B_n$ in $Y$ with respective centers $y_1, \ldots, y_n$. For $x \in B$, with $T_i x \in B_j$, $|T x - y_j| \leq |T x - T_i x| + |T_i x - y_j| < \varepsilon + \varepsilon$.

Thus, $TB$ is covered by a finite number of $2\varepsilon$-balls. This holds for every $\varepsilon > 0$, so $TB$ is totally bounded.

[2] In Banach spaces, the converse is false: there are compact operators which are not operator-norm limits of finite-rank operators. The counter-examples are difficult, due to Per Enflo.
Recall the proof that total boundedness of a set \( E \) in a complete metric space implies compact closure:

Since metric spaces have countable local bases, it suffices to show sequential compactness. That is, a sequence \( \{v_i\} \) in \( E \), exhibit a convergent subsequence.

Cover \( E \) by finitely-many \( 2^{-1} \)-balls, choose one, call it \( B_1 \), with infinitely-many \( v_i \) in \( E \cap B_1 \), and let \( w_1 \) be one of those infinitely-many \( v_i \).

Next, cover \( E \) by finitely-many \( 2^{-2} \)-balls. Certainly \( E \cap B_1 \) is covered by these, and \( E \cap B_1 \cap B_2 \) contains infinitely-many \( v_i \) for at least one of these, call it \( B_2 \). Let \( w_2 \in E \cap B_1 \cap B_2 \) be one of these \( v_i \), other than \( w_1 \).

Inductively, find an infinite subsequence \( w_n \) of distinct points, with \( w_n \in E \cap B_1 \cap \ldots \cap B_n \), where \( B_n \) is of radius \( 2^{-n} \). The sequence \( w_i \) is Cauchy. 

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