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Compact resolvents

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Unbounded operators T with compact resolvents $(T-\lambda)^{-1}$ are among the most useful among unbounded operators on Hilbert or Banach spaces. Many important semi-bounded symmetric *differential* operators are in this class, the simplest being (*regular*) Sturm-Liouville operators like $T = \frac{d^2}{dx^2} + q(x)$ on a finite interval $[a, b]$.

We prove that, for $T : X \to X$ a possibly unbounded, but densely-defined, operator on a Banach space, T^{-1} *compact* implies that the resolvent $(T-\lambda)^{-1}$ is *meromorphic*, and is *compact* away from poles. This is an example of *perturbation theory*.

The proof uses basic facts about compact operators. The easier case of T a *symmetric* operator on a Hilbert space is already useful. In that case, T^{-1} is a *normal* compact operator, and the resolvent $(T-\lambda)^{-1}$ is *normal*, allowing application of simple results about normal compact operators on Hilbert spaces, recalled in an appendix.

A fuller version of the spectral theory of compact operators on Banach spaces circumvents issues of *normality* and of the *symmetry* of T, and extends the discussion of compact resolvents to Banach spaces. The required ideas are *Fredholm-Riesz* theory, from [Fredholm 1900/1903] and [Riesz 1917].

The general Banach space setting is useful, directly addressing intuitive spaces such as $C^o[a, b]$ or $C^k[a, b]$.

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1. *Application of perturbation theory*

We prove that, if a (not necessarily bounded) densely-defined operator T on a Banach space X has *compact* inverse T^{-1}, then $(T-\lambda)^{-1}$ exists and is compact for λ off a *discrete* set in $\mathbb{C}$, and is *meromorphic* in λ.

The background on compact operators is more elementary in the interesting sub-case that T is a (not necessarily bounded) *symmetric* operator on a *Hilbert* space X. When T^{-1} exists and is *compact*, it is also *normal*, and the *normal* operator $(T-\lambda)^{-1}$ exists and is compact for λ off a *discrete* set in $\mathbb{C}$, and is *meromorphic* in λ.

The assertion and argument are standard, especially for Hilbert spaces. E.g., see [Kato 1966], p. 187 and preceding.

[1.1] Spectrum of possibly-unbounded operators

Recall that specification of a possibly unbounded operator T on a Hilbert or Banach space X includes its domain D_T. We only consider T with D_T *dense*.

The set of *eigenvalues* or *point spectrum* of a possibly-unbounded operator T consists of $\lambda \in \mathbb{C}$ such that $T-\lambda$ fails to be *injective*.

The *continuous* spectrum consists of λ with $T-\lambda$ *injective* and with *dense* image, but *not surjective*. Further, for possibly unbounded operators, we require a *bounded* (=continuous) inverse $(T-\lambda)^{-1}$ on $(T-\lambda)D_T$ for λ to be in the continuous spectrum.

The *residual spectrum* consists of λ with $T-\lambda$ injective, but $(T-\lambda)D_T$ not dense.

The description of *continuous spectrum* simplifies for *closed* T: we claim that for $(T-\lambda)^{-1}$ densely defined and continuous, $(T-\lambda)D_T$ is the whole space, so $(T-\lambda)^{-1}$ is *everywhere* defined, so λ cannot be in the residual spectrum. Indeed, the continuity gives a constant C such that $|x| \le C \cdot |(T-\lambda)x|$ for all $x \in D_T$. Then $(T-\lambda)x_i$ Cauchy implies x_i Cauchy, and T closed implies $T(\lim x_i) = \lim Tx_i$. Thus, $(T-\lambda)D_T$ is *closed*. Then *density* of $(T-\lambda)D_T$ implies it is the whole space.

[1.2] $(T-\lambda)^{-1}$ is compact

Now prove that for T^{-1} compact on a Banach space the resolvent $(T-\lambda)^{-1}$ exists and is compact for λ off a discrete set, and is meromorphic in λ.

The non-zero spectrum of the compact operator T^{-1} is *point spectrum*, from basic Fredholm-Riesz theory for compact operators. [1]

We claim that the spectrum of T and non-zero spectrum of T^{-1} are in the obvious bijection $\lambda \leftrightarrow \lambda^{-1}$. From the algebraic identities

$$T^{-1} - \lambda^{-1} \;=\; T^{-1}(\lambda - T)\lambda^{-1} \qquad\qquad T - \lambda \;=\; T(\lambda^{-1} - T^{-1})\lambda$$

failure of either $T-\lambda$ or $T^{-1}-\lambda^{-1}$ to be *injective* forces the failure of the other, so the point spectra are identical.

For (non-zero) λ^{-1} not an eigenvalue of *compact* T^{-1}, $T^{-1}-\lambda^{-1}$ is *bijective*. by Fredholm-Riesz theory. [2] Thus, $T^{-1}-\lambda^{-1}$ has a continuous, everywhere-defined inverse. For such λ, inverting $T-\lambda = T(\lambda^{-1}-T^{-1})\lambda$ gives

$$(T-\lambda)^{-1} \;=\; \lambda^{-1}(\lambda^{-1}-T^{-1})^{-1}T^{-1}$$

from which $(T-\lambda)^{-1}$ is continuous and everywhere-defined. That is, λ is *not* in the spectrum of T. Finally, $\lambda = 0$ is not in the spectrum of T, because T^{-1} exists and is continuous. This establishes the bijection.

Thus, for T^{-1} compact, the spectrum of T is *countable*, with no accumulation point in $\mathbb{C}$. Letting $R_\lambda = (T-\lambda)^{-1}$, the resolvent relation

$$R_\lambda \;=\; (R_\lambda - R_0) + R_0 \;=\; (\lambda - 0)R_\lambda R_0 + R_0 \;=\; (\lambda R_\lambda + 1)\circ R_0$$

expresses R_λ as the composition of a continuous operator with a compact operator, proving its compactness. ///

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2. *Appendix: normal compact operators on Hilbert spaces*

We prove an easy special case of a more general fact. Our result is that, for a normal, compact operator $T: X \to X$ on a Hilbert space X, for $\lambda \ne 0$ not an eigenfunction, $(T-\lambda)X = X$.

[2.1] $\mathrm{Im}\,(T-\lambda)$ is closed for $\lambda \ne 0$

We claim that, for a compact operator $T: X \to X$ on a Hilbert space X, for $\lambda \ne 0$, the image $(T-\lambda)X$ of $T-\lambda$ is *closed*.

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[1] This is an easy part of Fredholm-Riesz theory. There is a simpler proof that non-zero spectrum is point spectrum for T a *symmetric* operator on a *Hilbert* space, since then the assumed-compact operator T^{-1} is *normal*. This easier discussion is recalled in an appendix.

[2] Again, for T^{-1} a *normal* operator on a *Hilbert* space, there is an easier argument for this bijection, as in the appendix.

To see this, let $(T-\lambda)x_n \to y$. First consider the situation that $\{x_n\}$ is *bounded*. Compactness of T yields a convergent subsequence of Tx_n, and we replace x_n by this subsequence. Then $-\lambda x_n = y - Tx_n$ converges to $y - \lim Tx_n$, so x_n is convergent to $x_o \in X$, since $\lambda \neq 0$, and $Tx_o = y$.

Next, when the distance of x_n from $\ker(T-\lambda)$ is bounded by b, write $x_n = x_n' + x_n''$ with $x_n'' \in \ker(T-\lambda)$ and $x_n' \in \ker(T-\lambda)^\perp$. Then

$$|(T-\lambda)x_n| \;=\; |(T-\lambda)x_n'| \;\leq\; |T-\lambda|\cdot b \;<\; \infty$$

That is, $(T-\lambda)x_n$ is bounded.

In general, let $X' = X/\ker(T-\lambda)$ and $q : X \to X'$ the quotient map. Then $T-\lambda$ factors through q, by some continuous $S : X' \to X$. There is also the canonical map $j : X' \to \ker(T-\lambda)^\perp$ so that $q \circ j$ is the identity on X'.

We claim that there is $\delta > 0$ such that $|S\xi| \geq \delta$ for $|\xi| = 1$ in X'. To see this, suppose $S\xi_n \to 0$. Then $(T-\lambda)j\xi_n \to 0$. Since $j\xi_n$ is bounded, we can replace it by a subsequence so that $Tj\xi_n$ is convergent. Then $-\lambda j\xi_n = Tj\xi_n$ is convergent, so $j\xi_n$ is convergent. Thus, ξ_n is convergent to some ξ_o, with $|\xi_o| = \lim|\xi_n| = 1$. Apparently, $S\xi_o = \lim S\xi_n = 0$, contradiction, proving that $|S\xi| \geq \delta > 0$ for $|\xi| = 1$. Returning to the main argument, suppose that $(T-\lambda)x_n \to y_o$. With $\xi_n = qx_n$, $S\xi_n \to y_o$, and $S(\xi_m - \xi_n) \to 0$. By the claim, $\xi_m - \xi_n \to 0$, so ξ_n is bounded. That is, the distance from x_n to $\ker(T-\lambda)$ is bounded, reducing to the previous case. ///

[2.2] Normal operators

For *normal* operators $T : X \to X$, compact or not, for λ not an eigenvalue, $T-\lambda$ has *dense image*. To see this, let y be in the orthogonal complement to the image. Then

$$0 \;=\; \langle (T-\lambda)x, y\rangle \;=\; \langle x, (T^*-\bar{\lambda})y\rangle \qquad\qquad (\text{for all } x \in X)$$

Thus, $(T^*-\bar{\lambda})y = 0$. Then

$$|(T-\lambda)y|^2 \;=\; \langle (T-\lambda)y, (T-\lambda)y\rangle \;=\; \langle (T^*-\bar{\lambda})(T-\lambda)y, y\rangle \;=\; \langle (T-\lambda)(T^*-\bar{\lambda})y, y\rangle \;=\; 0$$

Since λ was not an eigenvalue, $y = 0$. ///

[2.2.1] Remark: Recall that the *residual spectrum* of a bounded operator $T : X \to X$ is the collection of λ such that $T-\lambda$ is injective but $(T-\lambda)X$ is not dense. Thus, the previous result asserts that (bounded) *normal operators have empty residual spectrum.*

[2.2.2] Corollary: For $0 \neq \lambda$ not an eigenvector of compact, normal T, $T-\lambda$ is *surjective.*

Proof: We saw that $(T-\lambda)X$ is *dense* for compact T and $\lambda \neq 0$ not an eigenvalue. For normal T that image is also *closed*, so must be the whole space. ///

[2.2.3] Remark: The *continuous* spectrum of a bounded operator T is λ with $T-\lambda$ *injective* and *dense image*, but not *closed* image. Thus, the corollary asserts that normal compact operators have empty non-zero continuous spectrum (and empty residual spectrum, as for any bounded, normal operator).

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