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Distributions supported on hyperplanes

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[0.0.1] **Theorem:** A distribution u on $\mathbb{R}^{m+n} \approx \mathbb{R}^m \times \mathbb{R}^n$ supported on $\mathbb{R}^m \times \{0\}$, is uniquely expressible as a locally finite sum of transverse differentiations followed by restriction and evaluations, namely, a locally finite sum

$$u \;=\; \sum_\alpha u_\alpha \;\circ\; \mathrm{res}^{\mathbb{R}^m \times \mathbb{R}^n}_{\mathbb{R}^m \times \{0\}} \;\circ\; D^\alpha$$

where α is summed over multi-indices $(\alpha_1, \ldots, \alpha_n)$, D^α is the corresponding differential operator on $\{0\} \times \mathbb{R}^n$, and u_α are distributions on $\mathbb{R}^m \times \{0\}$. Further,

$$\mathrm{spt}\, u_\alpha \;\times\; \{0\} \;\subset\; \mathrm{spt}\, u \qquad\qquad \text{(for all multi-indices } \alpha)$$

Proof: For brevity, let

$$\rho \;=\; \mathrm{res}^{R^m \times R^n}_{\mathbb{R}^m \times \{0\}} \;:\; C^\infty_c(\mathbb{R}^m \times \mathbb{R}^n) \;\longrightarrow\; C^\infty_c(\mathbb{R}^m)$$

be the natural restriction map of test functions on $\mathbb{R}^m \times \mathbb{R}^n$ to $\mathbb{R}^m \times \{0\}$, by

$$(\rho f)(x) \;=\; f(x, 0) \qquad\qquad \text{(for } x \in \mathbb{R}^m)$$

The adjoint $\rho^* : \mathcal{D}(\mathbb{R}^m) \to \mathcal{D}(\mathbb{R}^{m+n})$ is a continuous map of distributions on $\mathbb{R}^m$ to distributions on $\mathbb{R}^m \times \mathbb{R}^n$, defined by

$$(\rho^* u)(f) \;=\; u(\rho(f))$$

First, *if* we could apply u to functions of the form $F(x, y) = f(x) \cdot y^\beta$, and *if* u had an expression as a sum as in the statement of the theorem, then

$$u\big(f(x) \cdot \frac{y^\alpha}{\alpha!}\big) \;=\; (-1)^{|\beta|} \cdot u_\beta(f) \cdot \beta!$$

since most of the transverse derivatives evaluated at 0 vanish. This is not quite legitimate, since y^α is not a test function. However, we can take a test function ψ on $\mathbb{R}^n$ that is identically 1 near 0, and consider $\psi(y) \cdot y^\alpha$ instead of y^α, and reach the same conclusion.

Thus, if there *exists* such an expression for u, it is unique. Further, this computation suggests how to specify the u_α, namely,

$$u_\beta(f) \;=\; u\big(f(x) \otimes \frac{y^\beta}{\beta!} \cdot \psi(y) \cdot (-1)^{|\beta|}\big)$$

This would also show the containment of the supports.

Show that the sum of these u_β's does give u. Given an open U in $\mathbb{R}^{m+n}$ with compact closure, u on $\mathcal{D}_U$ has some finite order k. As a slight generalization of the fact that distributions supported on $\{0\}$ are finite linear combinations of Dirac delta and its derivatives, we have

[0.0.2] **Lemma:** Let v be a distribution of finite order k supported on a compact set K. For a test function φ whose derivatives up through order k vanish on K, $v(\varphi) = 0$. ///

For any test function $F(x, y)$,

$$\Phi(x, y) \;=\; F(x, y) \;-\; \sum_{|\alpha| \le k} (-1)^{|\alpha|} \, \frac{y^\alpha}{\alpha!} \, \psi(y) \, (D^\alpha F)(x, 0)$$

has all derivatives vanishing to order k on the closure of U. Thus, by the lemma, $u(\Phi) = 0$, which proves that u is equal to that sum, and also proves the local finiteness. ///