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ℓ^p *with* $0 < p < 1$ *is not locally convex*

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That is, with $0 < p < 1$, the topological vector space

$$\ell^p = \{\{x_i \in \mathbb{C}\} : \sum_i |x_i|^p < \infty\}$$

is not locally convex with the topology given by the metric $d(x, y) = |x - y|_p$ coming from

$$|x|_p = \sum_i |x_i|^p \qquad \text{(for } 0 < p < 1 \text{ no } p^{th} \text{ root!)}$$

It is *complete* with respect to this metric. Note that $|x|_p$ fails to be a *norm* by failing to be homogeneous of degree 1. The failure of local convexity is as follows.

Local convexity would require that the convex hull of the δ-ball at 0 be contained in some r-ball. That is, local convexity would require that, given δ, there is some r such that

$$\Big|\frac{1}{n} \cdot (\delta, 0, \ldots) + \ldots + \frac{1}{n} \cdot (\underbrace{0, \ldots, 0, \delta}_{n}, 0, \ldots)\Big|_p = \Big(\frac{\delta}{n}\Big)^p + \ldots + \Big(\frac{\delta}{n}\Big)^p < r \qquad \text{(for } n = 1, 2, 3, \ldots)$$

That is, local convexity would require that, given δ, there is r such that

$$n^{1-p} < \frac{r}{\delta^p} \qquad \text{(for } n = 1, 2, 3, \ldots)$$

This is impossible because $0 < p < 1$. ///

For contrast, to prove the triangle inequality for the alleged metric on ℓ^p with $0 < p < 1$, it suffices to prove that

$$(x + y)^p < x^p + y^p \qquad \text{(for } 0 < p < 1 \text{ and } x, y \geq 0)$$

To this end, take $x \geq y$. By the mean value theorem,

$$(x + y)^p \leq x^p + p\xi^{p-1} y \qquad \text{(for some } x \leq \xi \leq x + y)$$

and

$$x^p + p\xi^{p-1}y \leq x^p + px^{p-1}y \leq x^p + py^{p-1}y = x^p + py^p$$
$$\leq x^p + y^p \qquad \text{(since } p - 1 < 0 \text{ and } \xi \geq x \geq y)$$

This proves the triangle inequality for $0 < p < 1$. ///
