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Functional analysis exercises 04

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Due Fri, 24 Feb 2013, preferably as PDF emailed to me.

[04.1] Let λ be a *non-zero* not-necessarily-continuous linear functional on a topological vector space V. Show that $\ker\lambda$ is *dense* if and only if λ is *not* continuous.

[04.2] By considering the poles and residues of the meromorphic family of distributions

$$u_s \;=\; \text{(integration-against) } |x|^s \cdot \log|x| \qquad \text{(with } x \in \mathbb{R}^2\text{)}$$

on $\mathbb{R}^2$, show that up to a constant $\log|x|$ is a fundamental solution for the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, that is,

$$\Delta \log|x| \;=\; \delta \cdot \text{(constant)} \qquad \text{(on } \mathbb{R}^2\text{)}$$

[04.3] On the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, let δ_1 be the Dirac delta at a point (x_1, y_1), and δ_2 the Dirac delta at another point (x_2, y_2). Let $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Determine the Fourier series for a solution u of $\Delta u = \delta_1 - \delta_2$, and observe that this Fourier series is *not* absolutely convergent pointwise, although it is in $H^{1-\varepsilon}(\mathbb{T}^2)$ for every $\varepsilon > 0$.

Show that the similar equation $\Delta u = \delta_1$ has *no* solution in $H^{-\infty}(\mathbb{T}^2)$.

Further suppose both points have *irrational slope*, in the sense that x_1/y_1 and x_2/y_2 are not rational. Show that u restricted to *rational lines* $L_\xi = \{(t, t\xi) : t \in \mathbb{R}\}/\mathbb{Z}^2$ (with $\xi \in \mathbb{Q}$) has a Fourier series in $H^{+\infty}(\mathbb{T}^1)$. Thus, these restrictions are C^∞.

In particular, this shows that a Fourier series in two variables can be very well-behaved along all rational circles, but not converge absolutely pointwise as a function of two variables. In particular, in this example, u has a *logarithmic singularity* at both the special points.

[04.4] Give the Hilbert space ℓ^2 the *weak* topology, that is, with the locally convex topology given by the (separating family of) seminorms

$$p_x(y) \;=\; |\langle x, y\rangle| \qquad \text{(for } x, y \in \ell^2\text{)}$$

Let $\{e_i\}$ be an orthonormal basis. Show that $e_i \to 0$ in the weak topology, although certainly not in the original, *strong* topology on ℓ^2.

[04.5] Show that the unit ball $B = \{v \in \ell^2 : |v| \le 1\}$ is *compact* in the weak topology. [1]

[04.6] Show that the *weak-dual topology* on $H^{-\infty}(\mathbb{T}^n) = \big(H^\infty(\mathbb{T}^n)\big)^*$ is strictly coarser than the colimit-of-Hilbert-spaces topology given by $H^{-\infty}(\mathbb{T}^n) = \text{colim}_s H^{-s}(\mathbb{T}^n)$, where each H^{-s} has its Hilbert-space topology.

[04.7] [2] The *Hilbert transform* H on the circle $\mathbb{T}$ is given at first perhaps only for $C^1(\mathbb{T})$ functions f, by a principal value integral

$$Hf(x) \;=\; \frac{1}{\pi}\int_{\mathbb{T}} \frac{f(y)\,dy}{e^{iy} - e^{ix}} \;=\; \lim_{\varepsilon\to 0^+}\int_{x+\varepsilon}^{x+2\pi-\varepsilon} \frac{f(y)\,dy}{e^{iy} - e^{ix}}$$

[1] This is a very special case of the *Banach-Alaoglu* theorem for duals of locally convex topological vector spaces.

[2] For example, compare M. Taylor, *Pseudo Differential Operators*, SLN 416, Springer-Verlag, 1974, pp 4-5.

Verify that this operator is continuous *at least* as a map $H^\infty(\mathbb{T}) \to H^{-\infty}(\mathbb{T})$. Then determine the *Schwartz kernel* in $H^{-\infty}(\mathbb{T}^2)$ of the Hilbert transform. Show that it extends to a continuous map $L^2(\mathbb{T}) \to L^2(\mathbb{T})$.

In fact, show $H = 2P - 1$ where $P : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the orthogonal projection that kills off negative-index Fourier components:

$$P \; : \; \sum_{n\in\mathbb{Z}} c_n \, e^{inx} \;\longrightarrow\; \sum_{n\geq 0} c_n \, e^{inx}$$
