(*September 8, 2012*)

Functional analysis exercises 01

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Due Friday, 21 Sept 2012, preferably as PDF emailed to me.

[01.1] Exhibit a sequence in $C^o[a,b]$ converging to 0 in L^2-metric but not convergent in $C^o[a,b]$.

[01.2] Show that the unit ball

$$\{(x_1, x_2, \ldots) \in \ell^2 \;:\; \sum_n |x_n|^2 \le 1\}$$

is *not* compact, while the *Hilbert cube*

$$\{(x_1, x_2, \ldots) \in \ell^2 \;:\; |x_n| \le \frac{1}{n}\}$$

is compact, and generally, for a sequence $\varepsilon_n > 0$, the cube

$$\{(x_1, x_2, \ldots) \in \ell^2 \;:\; |x_n| \le \varepsilon_n\}$$

is compact if and only if $\sum_n \varepsilon_n^2 < \infty$.

[01.3] Construct a *dense* subset of $C^o[a,b]$ of *nowhere differentiable* functions, as follows. Let U_n be the subset of $C^o[a,b]$ consisting of f such that for *all* $t \in [a,b]$

$$|f(s) - f(t)| \;>\; n \cdot |s - t|$$

for *some* $s \in [a,b] \cap [t - \frac{1}{n}, t + \frac{1}{n}]$. Show U_n is *open*. Show that U_n is *dense*. Show $\bigcap_n U_n$ is *dense* and consists of *nowhere differentiable* continuous functions.

[01.4] *(Optional)* Let X, d be a complete metric space with no *isolated points*. [1] Show that any dense countable intersection of opens is *uncountable*. [2]

[01.5] Let E be the set of continuous functions $f \in C^o[0,2]$ with

$$\int_0^1 f\,dx - \int_1^2 f\,dx = 1$$

Show E is *closed* and *convex*, but has *no* element of minimal norm (in contrast to the Hilbert space situation).

[01.6] Let E be the functions f in $L^1[0,1]$ such that $\int_0^1 f\,dx = 1$. Show E is a closed convex subset with *infinitely many* elements of minimal norm (in contrast to the Hilbert space situation).

[1] A point $x \in X$ is *isolated* when there is $b > 0$ such that $d(x,y) \ge b$ for all $y \ne x$.

[2] The same argument proves the same uncountability result for *locally compact Hausdorff* spaces with no isolated points.