(*October 9, 2012*)

Functional analysis exercises 02

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is http://www.math.umn.edu/~garrett/m/fun/exercises_2012-13/fun-ex-10-05-2012.pdf]

Due Wed, 24 Oct 2012, preferably as PDF emailed to me.

[02.1] *Convincingly* and *not-ugly-ly* prove that e^{-1/x^2} (naturally extended by 0 at $x = 0$) is infinitely differentiable at 0.

[02.2] Show that for $0 \le c_n \in \mathbb{R}$ with c_n decreasing *monotonicly* to 0, the Fourier series $\sum_n c_n\, e^{inx}$ *converges* at $x \notin 2\pi\mathbb{Z}$, although not necessarily *absolutely*.

[02.3] *(Fejér kernel for $S^1 \times S^1$)* Prove completeness of suitable exponentials in $L^2(S^1 \times S^1)$ or higher dimensions directly, by constructing an *approximate identity* consisting of finite Fourier series. As expected, consider the *square* of the corresponding Dirichlet kernels

$$D_{M,N}(x,y) \;=\; \sum_{|m|\le M} \sum_{|n|\le N} e^{imx+iny} \;=\; \Big(\sum_{|m|\le M} e^{imx} \Big) \cdot \Big(\sum_{|n|\le N} e^{iny} \Big) \qquad \text{(Dirichlet kernel)}$$

The *Fejér kernel* is the normalization

$$\Phi_{M,N}(x,y) \;=\; \frac{D_{M,N}(x,y)^2}{\displaystyle\int_{S^1\times S^1} D_{M,N}(u,v)^2\; du\, dv} \qquad \text{(Fejér kernel)}$$

[02.4] Map $\mathbb{C}^n \to \mathbb{C}^{n+1}$ by $(z_1,\ldots,z_n) \to (z_1,\ldots,z_n,0)$. Let $V = \mathrm{colim}_n \mathbb{C}^n = \bigcup_n \mathbb{C}^n$, with the colimit topology, in which a basis of opens at 0 is given by convex hulls of unions $B = \bigcup_n B_n$ where B_n is an open ball of some positive radius, at 0, in $\mathbb{C}^n$. Show that V violates the conclusion of the Baire category theorem, so is not complete-metrizable. Here *Cauchy sequences* $\{x_n\}$ are those such that, given a neighborhood N of 0, there is n_o such that $x_m - x_n \in N$ for all $m, n \ge n_o$. Show that Cauchy sequences *converge*.

[02.5] Show that for complex w the equation $(\frac{d^2}{dx^2} - w^2)\, u_w = 0$ has a solution in $C^2(S^1)$ only when $w \in i\mathbb{Z}$. Let δ^{per} be the $2\pi\mathbb{Z}$ periodic Dirac δ-function, the continuous linear functional on $C^o(\mathbb{R}/2\pi\mathbb{Z})$ given by $\delta^{\mathrm{per}} f = f(0)$. With $w \in \mathbb{C}$ and $w \notin i\mathbb{Z}$, solve

$$\big(\frac{d^2}{dx^2} - w^2\big)\, u_w \;=\; \delta^{\mathrm{per}}$$

for u_w on $\mathbb{R}/2\pi\mathbb{Z}$. (*Hint:* expand δ^{per} in a Fourier series.) Identify the residues of the $L^2(S^1)$-valued meromorphic function $w \to u_w$.

[02.6] Show that there are *no eigenvectors* for the *Volterra operator* $T : L^2[0,1] \to L^2[0,1]$ given by

$$Tf(x) \;=\; \int_0^x f(y)\, dy$$

By design, $\frac{d}{dx} Tf = f$ for $f \in C^o[0,1]$. Show that $(T - \lambda)u = f$ is solvable for u when $0 \ne \lambda \in \mathbb{C}$ and $f \in C^1[0,1]$ with $f(0) = 0$, and the solution is unique. With T^* the Hilbert-space adjoint of T, show that

$$(TT^* f)(x) \;=\; \int_0^1 \min(x,y) \cdot f(y)\; dy \qquad\qquad (T^*Tf)(x) \;=\; \int_0^1 \big(1 - \max(x,y)\big) \cdot f(y)\; dy$$

Find eigenvectors for TT^* and T^*T.
