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Functional analysis exercises 03

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Due Wed, 12 Dec 2012, preferably as PDF emailed to me.

[03.1] Show that a compact subset of a metric space has a countable dense subset. (This makes it easy to show that every non-empty compact subset of $\mathbb{C}$ occurs as the spectrum of a multiplication operator on ℓ^2.)

[03.2] Show that the sup-norm completion of the space $C_c^o(\mathbb{R})$ of *compactly-supported* continuous functions is the space $C_o^o(\mathbb{R})$ of continuous functions f *vanishing at infinity*. (A function f *vanishes at infinity* if, for every $\varepsilon > 0$, there is compact $K \subset \mathbb{R}$ such that $|f(x)| < \varepsilon$ for $x \notin K$.)

[03.3] Determine the Fourier expansion, in two variables, of the Schwartz kernel for $\frac{d}{dx}$ on $C^\infty(S^1)$.

[03.4] Show that

$$\frac{d^2}{dx^2} \int_{\mathbb{R}} |x-y| \; f(y) \; dy \;=\; 2 \cdot f(x) \qquad \text{(for } f \in C_c^\infty(\mathbb{R})\text{)}$$

Apparently $K(x,y) = \frac{1}{2} \frac{\partial^2}{\partial x^2} |x-y|$ is a Schwartz kernel for the identity map $C_c^\infty(\mathbb{R}) C_c^\infty(\mathbb{R})$.

[03.5] Show that the *principal value* functional

$$\lambda(f) \;=\; \text{p.v.} \int_{\mathbb{R}} \frac{f(x)\; dx}{x} \;=\; \lim_{\varepsilon \to 0^+} \Big(\int_{-\infty}^{-\varepsilon} \frac{f(x)\; dx}{x} + \int_{\varepsilon}^{\infty} \frac{f(x)\; dx}{x} \Big) \qquad \text{(for } f \in C_c^\infty(\mathbb{R})\text{)}$$

is the derivative of integration-against $\log |x|$. That is, show that

$$\lambda(f) \;=\; -\int_{\mathbb{R}} f'(x) \cdot \log|x| \; dx$$

Compare $\lambda(f)$ to

$$\mu(f) \;=\; \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \frac{f(x)\; dx}{x + i\varepsilon}$$

[03.6] Take $n \geq 3$ to avoid some uninteresting complications. The function-valued function $s \to u_s$ defined by

$$u_s(x) \;=\; |x|^s \qquad \text{(for } \mathrm{Re}(s) > -n\text{, for } x \in \mathbb{R}^n\text{, and } s \in \mathbb{C}\text{)}$$

is holomorphic for $\mathrm{Re}(s) > -n$, and in that range $x \to u_s(x)$ is a locally integrable function, so gives a *distribution*. With the usual Laplacian $\Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$, compute Δu_s. Use this to meromorphically continue u_s to $s \in \mathbb{C}$ away from a discrete set of poles. Show that the *residue* at the right-most pole, at $s = -n$ is $C_n \cdot \delta$ for some non-zero scalar C_n. Use this to show that $\Delta u_{-n+2} = C_n \cdot \delta$. That is, up to a scalar, $|x|^{2-n}$ is a *fundamental solution* for Δ on $\mathbb{R}^n$.

[03.7] Let δ be the Dirac delta, and let H be the 'Heaviside function' which is 0 for $x < 0$ and 1 for $x > 0$. Let $\mathbf{1}$ be the identically-one function and $\mathbf{0}$ the identically-zero function. Show that

$$\delta \;=\; \delta' * H \qquad\qquad \mathbf{1} * \delta' \;=\; \mathbf{0}$$

and observe the failure of an alleged associativity:

$$(\mathbf{1} * \delta') * H \;=\; \mathbf{0} * H \;=\; \mathbf{0} \;\neq\; \mathbf{1} \;=\; \mathbf{1} * \delta \;=\; \mathbf{1} * (\delta' * H)$$
