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Fourier transforms, Fourier series, pseudo-Laplacians

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This is a simple application of spectral theory on a *homogeneous* ambient physical space to (what should be construed as) *boundary-valued problems* on an *inhomogeneous subset*.

This simple situation allows a direct computation *not* generally possible, seemingly ignoring the homogeneous ambient space. We first take advantage of this possibility, as a double-check on a second computation idea more generally available.

Here, the less direct method uses Fourier transform on $\mathbb{R}$ to decompose $L^2[a,b]$ with respect to Δ.

Here, the Friedrichs extension of a suitable restriction of Δ is easily understood.

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1. *From* $\mathbb{R}$ *to* $[a,b]$

[1.1] A Friedrichs' extension/pseudo-Laplacian $\tilde{\Delta}_{a,b}$ on $[a,b]$

For $a < b$,

$$L^2[a,b] \;=\; L^2\text{-completion of } C_c^\infty(a,b) \;=\; L^2(\mathbb{R}) \,\cap\, C_c^\infty(-\infty,a)^\perp \,\cap\, C_c^\infty(b,+\infty)^\perp$$

Let $\Delta_{a,b}$ be Δ restricted to

$$\text{domain } \Delta_{a,b} \;=\; C_c^\infty(a,b) \;=\; C_c^\infty(\mathbb{R}) \,\cap\, C_c^\infty(-\infty,a)^\perp \,\cap\, C_c^\infty(b,+\infty)^\perp$$

Note that test functions in $C_c^\infty(a,b)$ have support strictly inside the open interval (a,b). The vanishing condition at a,b and integration by parts prove *symmetry*

$$\langle \Delta_{a,b} v, w\rangle \;=\; \langle v, \Delta_{a,b} w\rangle \qquad (\text{for } v,w \in C_c^\infty(a,b))$$

The self-adjoint *Friedrichs extension* $\tilde{\Delta}_{a,b}$ of the symmetric operator $\Delta_{a,b}$ is defined as follows. Being a self-adjoint extension of a symmetric restriction of a Laplacian, it is a *pseudo-Laplacian.*

Define an inner product $\langle,\rangle_1$ whose associated norm dominates that of $\langle,\rangle$ by

$$\langle v,w\rangle_1 \;=\; \langle (1-\Delta_{a,b})v, w\rangle \;=\; \langle (1-\Delta)v, w\rangle \;=\; \langle v, (1-\Delta)w\rangle \qquad (\text{for } v,w \in C_c^\infty(a,b))$$

Let V_1 be the completion of $C_c^\infty(a,b)$ with respect to the associated norm. [1] The value $(1-\tilde{\Delta}_{a,b})^{-1}$ of the *resolvent* of the Friedrichs extension is defined via Riesz-Fischer on V_1: for $w \in L^2[a,b]$, the continuous linear functional $v \to \langle v,w\rangle$ on $v \in V_1$ is $v \to \langle v,w'\rangle_1$ for some $w' \in V_1$. The map $w \to w'$ is the value of the resolvent: $w' = (1-\tilde{\Delta}_{a,b})^{-1} w$. That is,

$$\langle v,w\rangle \;=\; \langle v,w'\rangle_1 \;=\; \langle v, (1-\tilde{\Delta}_{a,b})^{-1}w\rangle_1 \;=\; \langle (1-\Delta)v, (1-\tilde{\Delta}_{a,b})^{-1}w\rangle \qquad (v \in C_c^\infty(a,b) \text{ and } w \in L^2[a,b])$$

Density of V_1 in $L^2[a,b]$ proves well-definedness, etc.

Other boundary conditions than $u(a) = 0 = u(b)$ yield a different Friedrichs extension.

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[1] The norm $|f|_1 = \langle f,f\rangle_1^{\frac{1}{2}}$ is an example and antecedent of *Sobolev norms*, and the completion V_1 is an example and antecedent of a *Sobolev space*.

[1.2] Distributional nature of $\tilde{\Delta}_{a,b}$ The relation

$$\langle v, w\rangle \;=\; \langle (1-\Delta)v, (1-\tilde{\Delta}_{a,b})^{-1}w\rangle \qquad (v \in C_c^\infty(a,b) \text{ and } w \in L^2[a,b])$$

gives the same relation for the pairing $\langle,\rangle : C_c^\infty(a,b) \times C_c^\infty(a,b)^* \to \mathbb{C}$ obtained by applying a forgetful functor and treating $(1-\tilde{\Delta}_{a,b})^{-1}w$ as a *distribution*. Then

$$\langle v, w\rangle \;=\; \langle (1-\Delta)v, (1-\tilde{\Delta}_{a,b})^{-1}w\rangle \;=\; \langle v, (1-\Delta)(1-\tilde{\Delta}_{a,b})^{-1}w\rangle \qquad (v \in C_c^\infty(a,b) \text{ and } w \in L^2[a,b])$$

Thus,

$$0 \;=\; \langle v, w - (1-\Delta)(1-\tilde{\Delta}_{a,b})^{-1}w\rangle \;=\; \langle v, (1-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w - (1-\Delta)(1-\tilde{\Delta}_{a,b})^{-1}w\rangle$$
$$=\; \langle v, (\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w\rangle \qquad (v \in C_c^\infty(a,b) \text{ and } w \in L^2[a,b])$$

That is, the support of $(\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w$ does not meet the open interval (a,b). On the other hand, the support of the *distribution* $(\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w$ is at most $[a,b]$. Thus,

$$\text{spt}\Big((\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w\Big) \;\subset\; \{a,b\}$$

By its definition, V_1 is contained in the $+1$-index *Sobolev space* $H^1(\mathbb{R})$ on $\mathbb{R}$. Since $(1-\tilde{\Delta}_{a,b})^{-1}$ maps $L^2[a,b]$ to V_1, and Δ is degree two, $(\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w$ is at worst in $H^{-1}(\mathbb{R})$. By classification of distributions supported on isolated points, distributions supported on $\{a,b\}$ are finite linear combinations of derivatives of δ_a, δ_b, the Dirac deltas at a, b. The deltas themselves are in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$, but all higher derivatives are outside $H^{-1}(\mathbb{R})$.

Therefore,

$$(\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w \;\in\; \mathbb{C}\cdot\delta_a \oplus \mathbb{C}\cdot\delta_b \qquad (\text{for all } w \in L^2[a,b])$$

Conversely, when $(\Delta-\tilde{\Delta}_{a,b})(1-\tilde{\Delta}_{a,b})^{-1}w$ is such a linear combination, it pairs to 0 against $v \in C_c^\infty(a,b)$.

Thus, $\tilde{\Delta}_{a,b}$ differs from Δ by ignoring $\mathbb{C}\cdot\delta_a \oplus \mathbb{C}\cdot\delta_b$. A function $u \in L^2(\mathbb{R})_{a,b}$ is a λ-eigenfunction for $\tilde{\Delta}_{a,b}$ if and only if

$$(\Delta-\lambda)u \;\in\; \mathbb{C}\cdot\delta_a + \mathbb{C}\cdot\delta_b \qquad (\text{distributional derivative})$$

[1.3] Truncations and eigenfunctions: cheating The elementary nature of the situation allows short-cuts in solving differential equations that would not be generally available. We take this route first, although it does not suggest general methods.

Let $\lambda = \lambda_w = w^2$. In the interior of $[a,b]$, a $\tilde{\Delta}_{a,b}$-eigenfunction is a *genuine* Δ-eigenfunction locally on the *interior* of $[a,b]$, so is a linear combination of solutions of $(\Delta - w^2)u = 0$ there.

This differential equation has elementary solutions $u(x) = e^{\pm wx}$. We take advantage of this to simplify the computation.

However, the exponentials $e^{\pm wx}$ on $\mathbb{R}$ do not vanish outside $[a,b]$. That is, these exponentials are eigenfunctions on the *ambient* space $\mathbb{R}$. We might try to use *truncations* of these eigenfunctions to $[a,b]$:

$$\tau f(x) \;=\; \begin{cases} f(x) & (\text{for } x \in [a,b]) \\ 0 & (\text{for } x \text{ outside } [a,b]) \end{cases} \;=\; f(x)\cdot\big(H_a(x) - H_b(x)\big)$$

where $H_c(x)$ is the Heaviside step function with values 0 for $x < c$ and 1 for $x > c$. At the endpoints a, b, application of $\Delta - w^2$ to τe^{wx} will not annihilate it, but will leave distributions supported at a, b:

$$(\Delta - w^2)(\tau e^{wx}) \;=\; (\frac{d}{dx} + w)\big(e^{wx}(\delta_a - \delta_b)\big) \;=\; e^{wa}\cdot(\delta_a' + w\delta_a) - e^{wb}\cdot(\delta_b' - w\delta_b)$$

We want a linear combination $u(x) = A\tau e^{wx} + B\tau e^{-wx}$ such that $(\Delta - w^2)u$ has no δ' terms. This is equivalent to having a non-trivial solution A, B of

$$\begin{cases} Ae^{wa} + Be^{-wa} = 0 \\ Ae^{wb} + Be^{-wb} = 0 \end{cases}$$

which is equivalent to the vanishing

$$0 = \det \begin{bmatrix} e^{wa} & e^{-wa} \\ e^{wb} & e^{-wb} \end{bmatrix} = e^{(a-b)w} - e^{-(a-b)w}$$

The same condition is found by noting that, by construction, eigenfunctions for a Friedrichs extension lie in V_1, which in $\mathbb{R}$ lies inside C^o, by Sobolev's inequalities/imbedding. That is, truncations must not cause discontinuities. Thus, we require

$$\begin{cases} \delta_a\Big(\tau(Ae^{wx} + Be^{-wx}\Big) = 0 \\ \delta_b\Big(\tau(Ae^{wx} + Be^{-wx}\Big) = 0 \end{cases}$$

which gives the same system

$$\begin{cases} Ae^{wa} + Be^{-wa} = 0 \\ Ae^{wb} + Be^{-wb} = 0 \end{cases}$$

This is $e^{2(a-b)w} = 1$, or $e^{2aw} = e^{2wb}$. In that case, the two equations in A, B are linearly dependent, and either can be solved to express one of A, B in terms of the other. For example, with $A = 1$, $B = -e^{2wa} = -e^{2wb}$. Then

$$u_w(x) = Ae^{wx} + Be^{-wx} = e^{wx} - e^{2wa}\, e^{-wx} = e^{wx} - e^{2wb}\, e^{-wx}$$

With hindsight, multiplying by e^{-wa} (respectively, e^{-wb}) gives a more symmetrical expression

$$u_w(x) = e^{w(x-a)} - e^{-w(x-a)} = \pm\Big(e^{w(x-b)} - e^{-w(x-b)}\Big) \qquad \text{(eigenfunction for } w \in \tfrac{\pi i}{a-b}\mathbb{Z})$$

The first expression demonstrates $u(a) = 0$, the second that $u(b) = 0$, so neither δ'_a nor δ'_b occurs in the second derivative. In summary, for $w \in \frac{\pi i}{a-b}\mathbb{Z}$ the restriction/truncation of u_w to $[a, b]$ is an eigenfunction for $\tilde{\Delta}_{a,b}$ with eigenvalue w^2.

[1.4] Application of Fourier transform

Harmonic analysis on the ambient homogeneous space $\mathbb{R}$ applies to eigenfunctions for $\tilde{\Delta}_{a,b}$ on the inhomogeneous subset $[a, b]$. A necessary and sufficient condition for w^2 to be an eigenvalue is determined by Fourier transform methods.

An eigenfunction u_w for $\tilde{\Delta}_{a,b}$ is in $L^2[a, b] \subset L^2(\mathbb{R})$, so has a Fourier-Plancherel inverse-transform representation on $\mathbb{R}$:

$$u_w = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x}\, c(\xi)\, d\xi \qquad \text{(equality in an } L^2(\mathbb{R}) \text{ sense)}$$

A linear combination $A\delta_a + B\delta_b$ has such a representation

$$A\delta_a + B\delta_b = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x}\, \big(Ae^{-i\xi a} + Be^{-i\xi b}\big)\, d\xi \qquad (\text{in } H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}))$$

On the spectral side, application of $\Delta - w^2$ produces

$$\int_{\mathbb{R}} e^{i\xi x}\, (-\xi^2 - w^2) \cdot c(\xi)\, d\xi = \int_{\mathbb{R}} e^{i\xi x}\, \big(Ae^{-i\xi a} + Be^{-i\xi b}\big)\, d\xi$$

The desymmetrized Plancherel pairing on Sobolev spaces gives

$$(-\xi^2 - w^2) \cdot c(\xi) \;=\; Ae^{-i\xi a} + Be^{-i\xi b}$$

and then

$$c(\xi) \;=\; \frac{Ae^{-i\xi a} + Be^{-i\xi b}}{(-\xi^2 - w^2)} \qquad\qquad (\text{in an } L^2 \text{ sense})$$

For $c(\xi)$ to be in $L^2(\mathbb{R})$, necessarily $Ae^{-i\xi a} + Be^{-i\xi b}$ vanishes at both $\xi = \pm iw$:

$$\begin{cases} Ae^{wa} + Be^{wb} &= 0 \\ \\ Ae^{-wa} + Be^{-wb} &= 0 \end{cases}$$

Existence of a non-trivial solution is equivalent to $e^{2w(a-b)} = 1$, as earlier.

[1.5] Eigenvectors via fundamental solutions

Harmonic analysis on the ambient homogeneous space $\mathbb{R}$ can produce eigenfunctions for $\tilde{\Delta}_{a,b}$ on the inhomogeneous subset $[a, b]$.

Since $\tilde{\Delta}_{a,b}$ ignores δ_a and δ_b, we solve

$$(\Delta - w^2)u_w \;=\; A\delta_a + B\delta_b$$

on $\mathbb{R}$, and require $\delta_a u_w = 0$ and $\delta_b u_w = 0$ so that the *truncation* τu_w of u_w to $[a, b]$ is an eigenfunction for $\tilde{\Delta}_{a,b}$.

Solve $(\Delta - w^2)u_w = \delta_a$ and $(\Delta - w^2)u_w = \delta_b$ separately: under Fourier transform the first of these gives

$$(-\xi^2 - w^2) \cdot \widehat{u}_{w,a}(\xi) \;=\; e^{-ia\xi}$$

and

$$\widehat{u}_{w,a}(\xi) \;=\; \frac{e^{-ia\xi}}{-(\xi^2 + w^2)}$$

and the *fundamental solution* is

$$u_{w,a}(x) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \cdot \frac{e^{-ia\xi}\, d\xi}{-(\xi^2 + w^2)} \;=\; \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\xi(x-a)}\, d\xi}{-(\xi^2 + w^2)} \;=\; \frac{e^{-w|x-a|}}{-2w}$$

by residues. We will minimize use of the explicit form of the fundamental solution. As expected, for $\mathrm{Re}(w) > 0$ the fundamental solution is rapidly decreasing. in fact, since $1/(\xi^2 + w^2)$ is in a Paley-Wiener space, the fundamental solution should be expected to be of exponential decay depending on $\mathrm{Re}(w)$.

The denominator is *even*, so the integral can be rewritten

$$u_{w,a}(x) \;=\; \frac{1}{\pi} \int_{\mathbb{R}} \frac{\cos \xi(x-a)\, d\xi}{-(\xi^2 + w^2)}$$

For $\mathrm{Re}(w) > 0$, the integral converges well, and in that region the smoothness of $1/(\xi^2 + w^2)$ implies that $u_w(x)$ is of *rapid decay.*

However, eigenvalues w^2 for the non-positive self-adjoint $\tilde{\Delta}_{a,b}$ will be *non-positive real*, so $w \in i\mathbb{R}$ are the interesting parameter values. To allow $\mathrm{Re}(w) \to 0^+$ without worry, for $\mathrm{Re}(w) > 0$ rewrite

$$\begin{aligned} u_{w,a}(x) \;&=\; \frac{1}{\pi} \int_{\mathbb{R}} \frac{\cos \xi(x-a) - \cos\big(\pm iw(x-a)\big)\, d\xi}{-(\xi^2 + w^2)} \;+\; \cos\big(\pm iw(x-a)\big) \cdot \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\xi}{-(\xi^2 + w^2)} \\ &=\; \frac{1}{\pi} \int_{\mathbb{R}} \frac{\cos \xi(x-a) - \cos\big(\pm iw(x-a)\big)\, d\xi}{-(\xi^2 + w^2)} \;-\; \frac{\cos\big(\pm iw(x-a)\big)}{w} \end{aligned}$$

[1.5.1] Remark: This gives a meromorphic continuation of $u_{w,a}$ to $\mathbb{C} - \{0\}$, but *not* as an $L^2(\mathbb{R})$ function. To make the meromorphic continuation effortless, we must allow the extension to take values in spaces of functions of *exponentional growth* as $|x| \to \infty$, depending on $\mathrm{Re}(w)$.

A similar computation applies to $u_{w,b}$. Application of δ_a, δ_b, which are in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{R})$, is justified since $u_{w,a} \in H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$. However, we use the explicit form of the fundamental solutions:

$$\delta_a u_{w,a} \;=\; \frac{-1}{2w} \qquad \delta_b u_{w,a} \;=\; \frac{e^{-w|b-a|}}{-2w} \qquad \delta_b u_{w,b} \;=\; \frac{-1}{2w} \qquad \delta_a u_{w,b} \;=\; \frac{e^{-w|b-a|}}{-2w}$$

The condition for $(\Delta - w^2)u_{w,A,B} = A\delta_a + B\delta_b$ to have a non-trivial solution with $\delta_a u_{w,A,B} = 0$ and $\delta_b u_{w,A,B} = 0$ is

$$0 \;=\; \det \begin{pmatrix} \frac{-1}{2w} & \frac{e^{-w|b-a|}}{-2w} \\ \frac{e^{-w|b-a|}}{-2w} & \frac{-1}{2w} \end{pmatrix} \;=\; \frac{1 - e^{-2w|b-a|}}{4w^2}$$

Thus, again we find the condition $w \in \frac{\pi i}{b-a}\mathbb{Z}$.

[1.5.2] Remark: We have a meromorphic continuation of $u_{w,A,B}$ to $\mathbb{C} - \{0\}$, *not* as an $L^2(\mathbb{R})$ function, but taking values in spaces of functions of *exponentional growth* as $|x| \to \infty$, depending on $\mathrm{Re}(w)$. Only as w approaches the *special* points $\frac{\pi i}{b-a}\mathbb{Z}$ from the right does the meromorphically-continued solution stay in $L^2(\mathbb{R})$. This is *not* readily visible from the integral expressions.

Bibliography

[CdV 1982,83] Y. Colin de Verdière, *Pseudo-laplaciens, I, II*, Ann. Inst. Fourier (Grenoble) **32** (1982) no. 3, 275-286, **33** (1983) no. 2, 87-113.

[Faddeev 1967] L. Faddeev, *Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobacevskii plane*, AMS Transl. Trudy (1967), 357-386.

[Faddeev-Pavlov 1972] L. Faddeev, B.S. Pavlov *Scattering theory and automorphic functions*, Seminar Steklov Math. Inst. **27** (1972), 161-193.

[Garrett 2011a] P. Garrett, *Colin de Verdière's meromorphic continuation of Eisenstein series*
http://www.math.umn.edu/~garrett/m/v/cdv_eis.pdf

[Garrett 2011b] P. Garrett, *Compact operators on Banach spaces: Fredholm-Riesz*
http://www.math.umn.edu/~garrett/m/fun/fredholm_riesz.pdf

[Garrett 2011c] P. Garrett, *Compact resolvents*
http://www.math.umn.edu/~garrett/m/fun/compact_resolvent.pdf

Garrett 2012] P. Garrett, *Compact operators, Hilbert-Schmidt operators*
http://www.math.umn.edu/~garrett/m/fun/compact_ops.pdf

[Grubb 2009] G. Grubb, *Distributions and operators*, Springer-Verlag, 2009.

[Kato 1966] T. Kato, *Perturbation theory for linear operators*, Springer, 1966, second edition, 1976, reprinted 1995.

[Lang 1975] S. Lang, $SL_2(\mathbb{R})$, Addison-Wesley, 1975.

[Lax-Phillips 1976] P. Lax, R. Phillips, *Scattering theory for automorphic functions*, Annals of Math. Studies, Princeton, 1976.

[Povzner 1953] A. Povzner, *On the expansion of arbitrary functions in characteristic functions of the operator* $-\Delta u + cu$, Math. Sb. **32** no. 74 (1953), 109-156.

[Riesz-Nagy 1952] F. Riesz, B. Szökefalvi.-Nagy, *Functional Analysis*, English translation, 1955, L. Boron, from *Lecons d'analyse fonctionelle* 1952, F. Ungar, New York, 1955